Dynamical Systems

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Lecture 2 of 5
Terminology recap

• Variable or state
• Differential equation
• Initial condition
• Trajectory
• Parameter
• Steady state
• Transient behaviour
• Perturbation
• Ordinary differential equations (ODE)
• 3 dimensional ODE
Solving for a steady state analytically

\[ \frac{dx}{dt} = k*x + c \]
\[ \frac{dx}{dt} = 0 \]

\[ k*x_{ss} + c = 0 \]
\[ k*x_{ss} = -c \]
\[ x_{ss} = \frac{-c}{k} \]
Population dynamics

dR/dt = k*R

R: Rabbit population
k: ?
Population dynamics

\[ \frac{dR}{dt} = k \cdot R \cdot (1-a) \]

- \( a=0? \)
- \( a=1? \)
- \( 0<a<1? \)
- \( a>1? \)
Feedback inhibition

[X] -> Protein X concentration
[Y] -> Protein Y is the dimer of XX

\[
\begin{align*}
\frac{d[X]}{dt} &= -k*X - r*X*X + (g-Y) \\
\frac{d[Y]}{dt} &= r*X*X - l*Y
\end{align*}
\]

k: ?, l: ?
r: ?
g: ?
Neural population model

E -> fractional firing of excitatory neural population
I -> fractional firing of inhibitory neural population

dE/dt = -E + S(a*E - b*I + P)
dI/dt = -I + S(c*E - d*I + Q)

a,b,c,d: connectivity weights
P,Q: baseline input to the populations
S: sigmoid function

Also known as Wilson-Cowan equations
Overview

• What are dynamical systems?
• How to interpret a differential equation
• **How to analyse differential equation systems**
• How to solve differential equation systems
• Stability analysis, multistability
• Oscillatory solutions
• Parameter variations, bifurcations
• Choice of cool stuff: Chaos, turbulence, spatio-temporal systems, slow-fast systems, transients, and more.
Time series 2
An alternative view: Phase space
Phase space with more initial conditions
Phase space with vector field
Vectorfield 101

dX/dt=2*Y-X

dY/dt=-Y+1

Draw phase space and vectorfield between X=[0:3], Y=[0:3]
Example of vector field and trajectory

http://earth.nullschool.net/#current/wind/surface/level/orthographic=-1.68,54.81,2461
Neural population model

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\frac{dE}{dt} &= -E + S(aE - bI + P) \\
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\end{align*}
\]

Also known as Wilson-Cowan equations

a, b, c, d: connectivity weights
P, Q: baseline input to the populations
S: sigmoid function
Phase space with vector field
Phase space with vector field and nullcline
Increase self excitation (parameter $a$)
Stable focus

Saddle

Stable node
Illustrating fixed point stability
Focus vs. node
Another example system: Feedback inhibition

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\frac{d[Y]}{dt} &= r*X*X - l*Y
\end{align*}
\]

k=0.1: degradation rate of X
l=0.8: degradation rate of Y
r=0.5: dimerisation rate
g=2: production rate of X
Time series
Phase space
Trajectories
Terminology recap

- Phase space/state space
- Vectorfield
- Fixed point (stable/unstable, focus/node)
- Nullcline
- Saddles, separatrix
- Bistability
Plotting phase space, vectorfields, nullclines

Google: pplane matlab

http://math.rice.edu/~dfield/#8.0
Derive nullclines

dX/dt = -k*X - r*X^2 + g - Y

dY/dt = r*X^2 - l*Y