Dynamical Systems

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Lecture 2 of 8
Overview

• What are dynamical systems?
• How to interpret a differential equation
• How to analyse differential equation systems
• How to solve differential equation systems
• Stability analysis, multistability
• Oscillatory solutions
• Parameter variations, bifurcations
• Choice of cool stuff: Chaos, turbulence, spatio-temporal systems, slow-fast systems, transients, and more.
Simple enzyme reaction: Dimerisation

[A] -> molecule A concentration

[AA] -> molecule AA (dimer) concentration

\[
\frac{d[A]}{dt} = -r \cdot A \cdot A
\]

\[
\frac{d[AA]}{dt} = +r \cdot A \cdot A
\]

r -> dimerisation rate
Enzyme catalysed reaction

[ADP] -> ADP concentration
[ATP] -> ATP concentration
[ATPS] -> ATP synthase concentration

\[
\frac{d[ADP]}{dt} = -r*[ATPS]*[ADP]
\]
\[
\frac{d[ATP]}{dt} = +r*[ATPS]*[ADP]
\]
\[
\frac{d[ATPS]}{dt} = c - k*[ATPS]
\]

r -> ATP conversion rate

\[
c -> ATPS transcription/translation rate
\]

k -> ATPS degradation rate
Feedback inhibition

[X] -> Protein X concentration
[Y] -> Protein Y is the dimer of XX

\[ \frac{d[X]}{dt} = -k*X -r*X*X + (g-Y) \]
\[ \frac{d[Y]}{dt} = r*X*X - l*Y \]

k: degradation rate of X, l: degradation rate of Y
r: dimerisation rate
g: production rate of X
Enzyme conversion: Michaelis Menten kinetics

[S] -> Substrate concentration
[P] -> Product concentration

d[S]/dt = -d[P]/dt

\[
\frac{d[P]}{dt} = \frac{(V_{\text{max}}[S])}{(K_m + [S])}
\]

Vmax: maximum conversion speed
Km: concentration of [S] at which the conversion speed is half of Vmax
A remark on terminology

• \( \frac{dx}{dt} = f(x) \) is called an ordinary differential equation.
• \( \frac{dx}{dt} = f(x,y), \frac{dy}{dt} = g(x,y) \) is called a 2 dimensional ordinary differential equation system.
• \( \frac{dx_1}{dt} = f(x_1, x_2, x_3, \ldots, x_n), \frac{dx_2}{dt} = g(x_1, x_2, x_3, \ldots, x_n) \ldots \) is called a \( n \) dimensional ordinary differential equation system.
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Neural population model

E -> fractional firing of excitatory neural population
I -> fractional firing of inhibitory neural population

dE/dt = -E + S(a*E - b*I + P)  
dI/dt = -I + S(c*E - d*I + Q)

a,b,c,d: connectivity weights
P,Q: baseline input to the populations
S: sigmoid function

Also known as Wilson-Cowan equations
Sigmoid: neuronal firing cannot increase indefinitely

\[
\text{Sigm}(x) = \frac{1}{1 + \exp(-s(x-o))}
\]

- \( s = 1 \) steepness
- \( o = 4 \) offset
Time series

- **Excitatory population**
- **Inhibitory population**
Time series 2
An alternative view: Phase space
Phase space with more initial conditions
Phase space with vector field
Terminology recap

- Dimensionality of a differential equation system
- ODE
- Phase space/state space
- Vectorfield
- Fixed point