Dynamical Systems

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Lecture 2 of 5

Terminology recap

- Variable or state
- Differential equation
- Initial condition
- Trajectory
- Parameter
- Steady state
- Transient behaviour
- Perturbation
- Ordinary differential equations (ODE)
- 3 dimensional ODE

Population dynamics

$$dR/dt = k*R$$

R: Rabbit population

k: ?

Population dynamics

$$dR/dt = k*R*(1-a)$$

$a=0$?

$a=1$?

$0<a<1$?

$a>1$?

Neural population model

E -> fractional firing of excitatory neural population

I -> fractional firing of inhibitory neural population

$$dE/dt = -E + S(a*E - b*I + P)$$

$$dI/dt = -I + S(c*E - d*I + Q)$$

Also known as Wilson-Cowan equations

a,b,c,d: connectivity weights

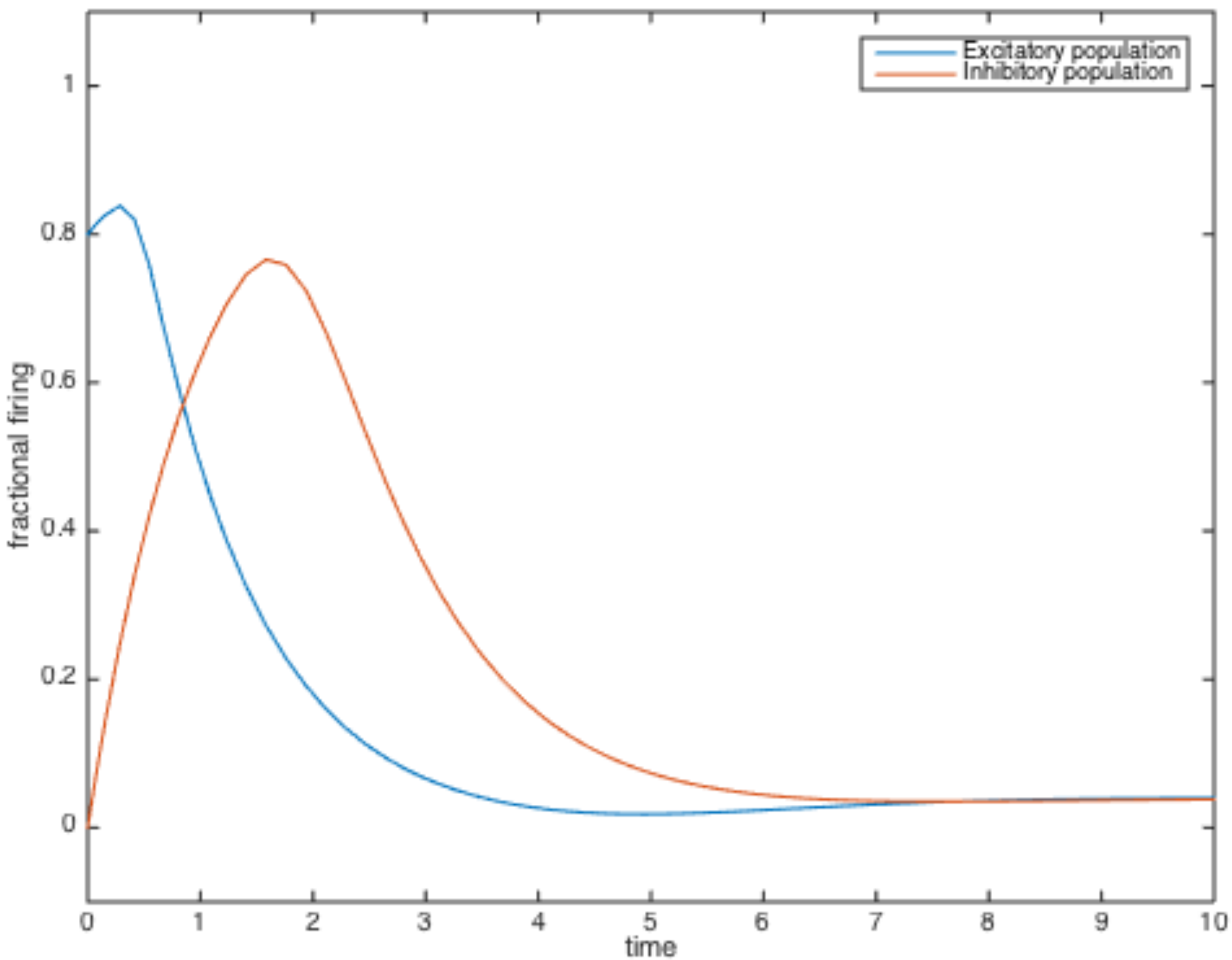
P,Q: baseline input to the populations

S: sigmoid function

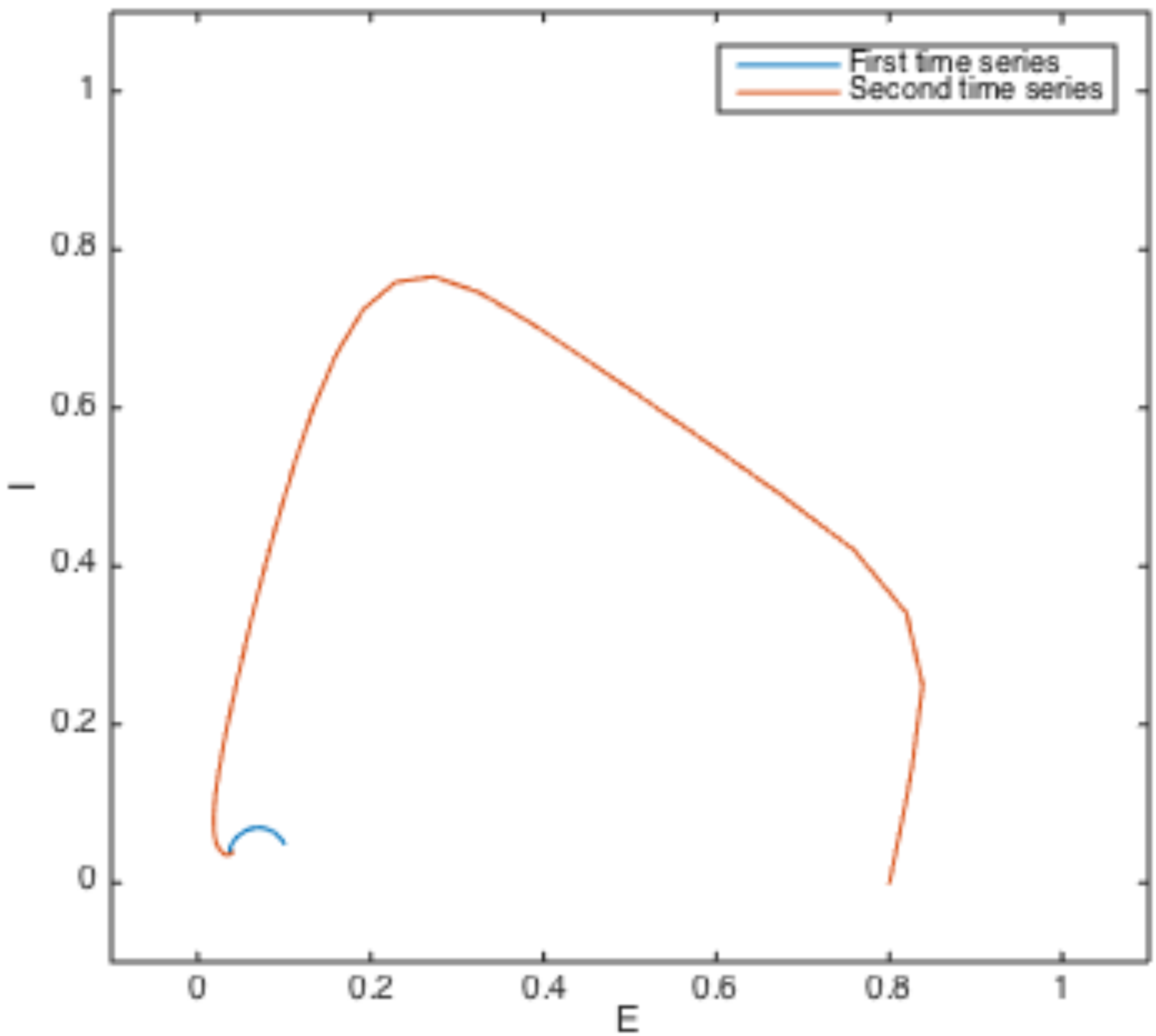
Overview

- What are dynamical systems?
- How to interpret a differential equation
- How to analyse differential equation systems
- How to solve differential equation systems
- Stability analysis, multistability
- Oscillatory solutions
- Parameter variations, bifurcations
- Choice of cool stuff: Chaos, turbulence, spatio-temporal systems, slow-fast systems, transients, and more.

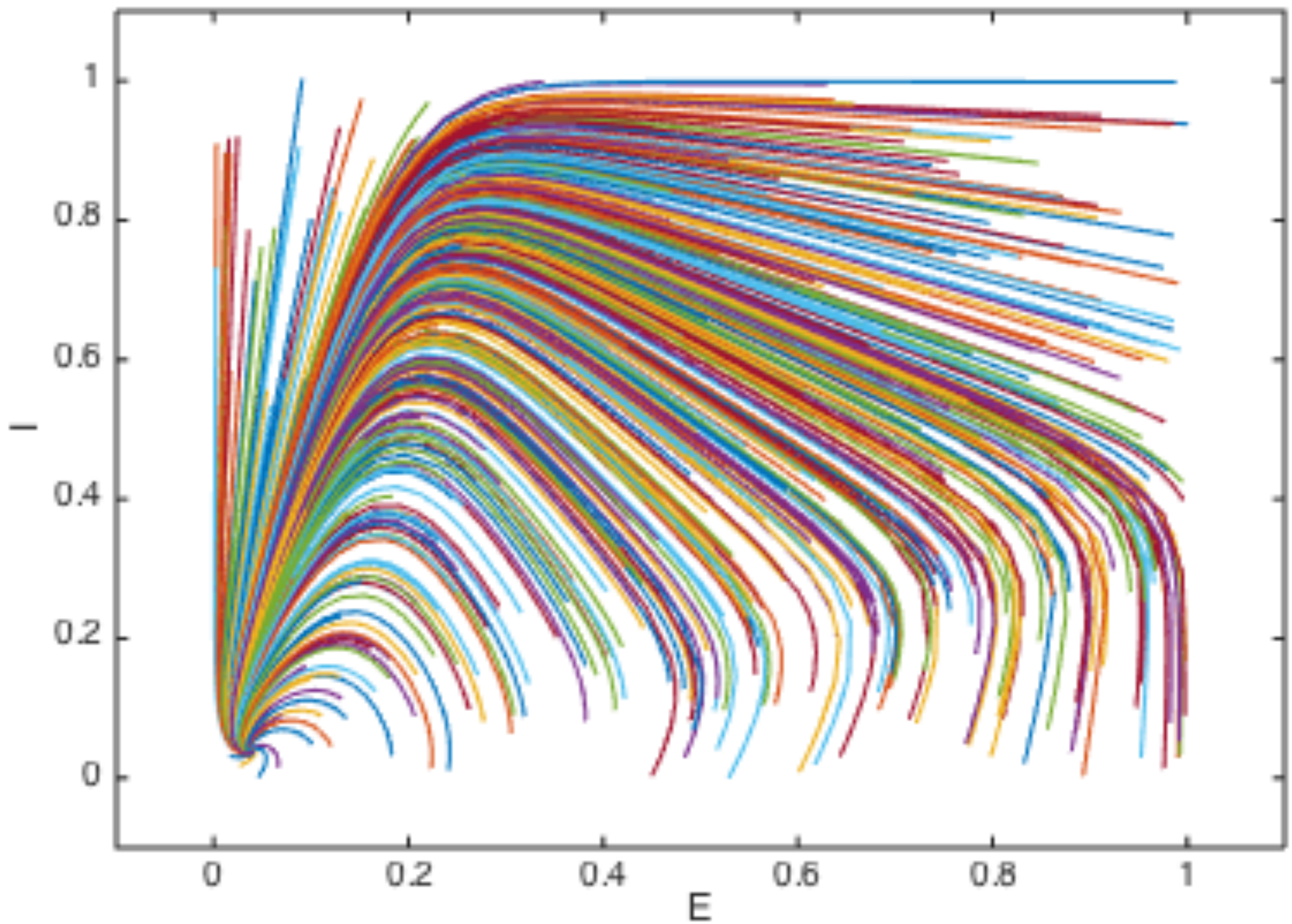
Time series 2



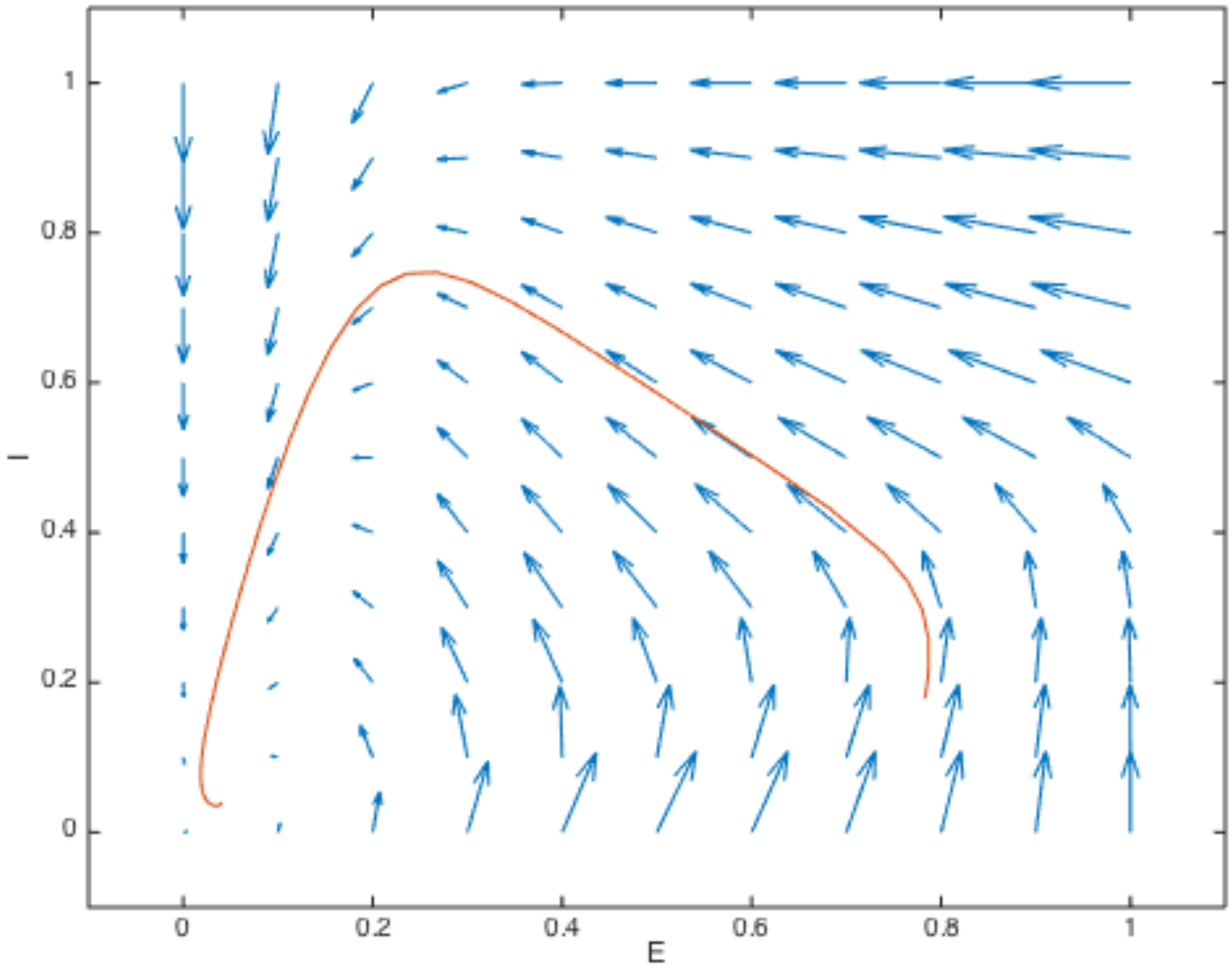
An alternative view: Phase space



Phase space with more initial conditions



Phase space with vector field



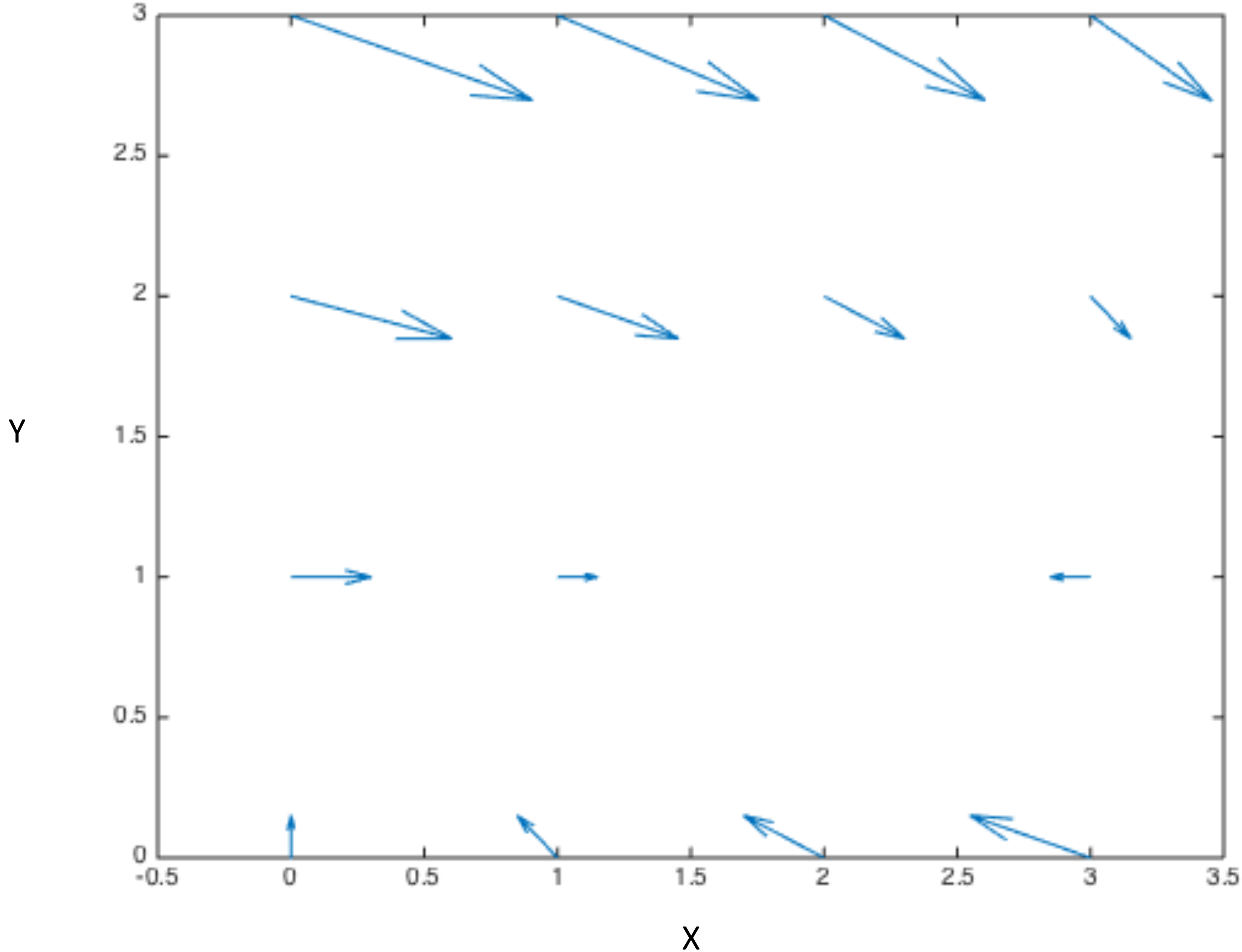
Vectorfield 101

$$dX/dt=2*Y-X$$

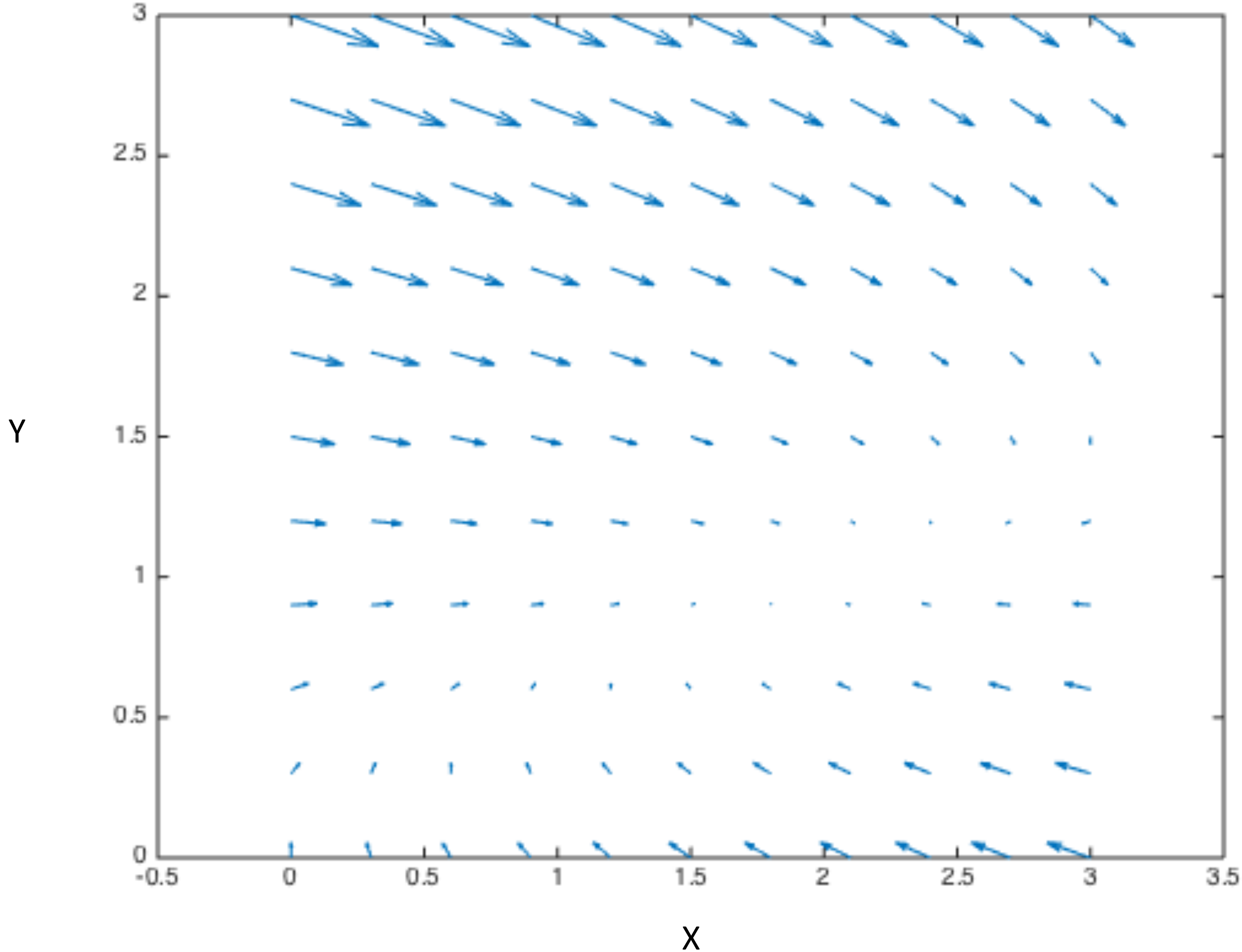
$$dY/dt=-Y+1$$

Draw phase space and vectorfield between

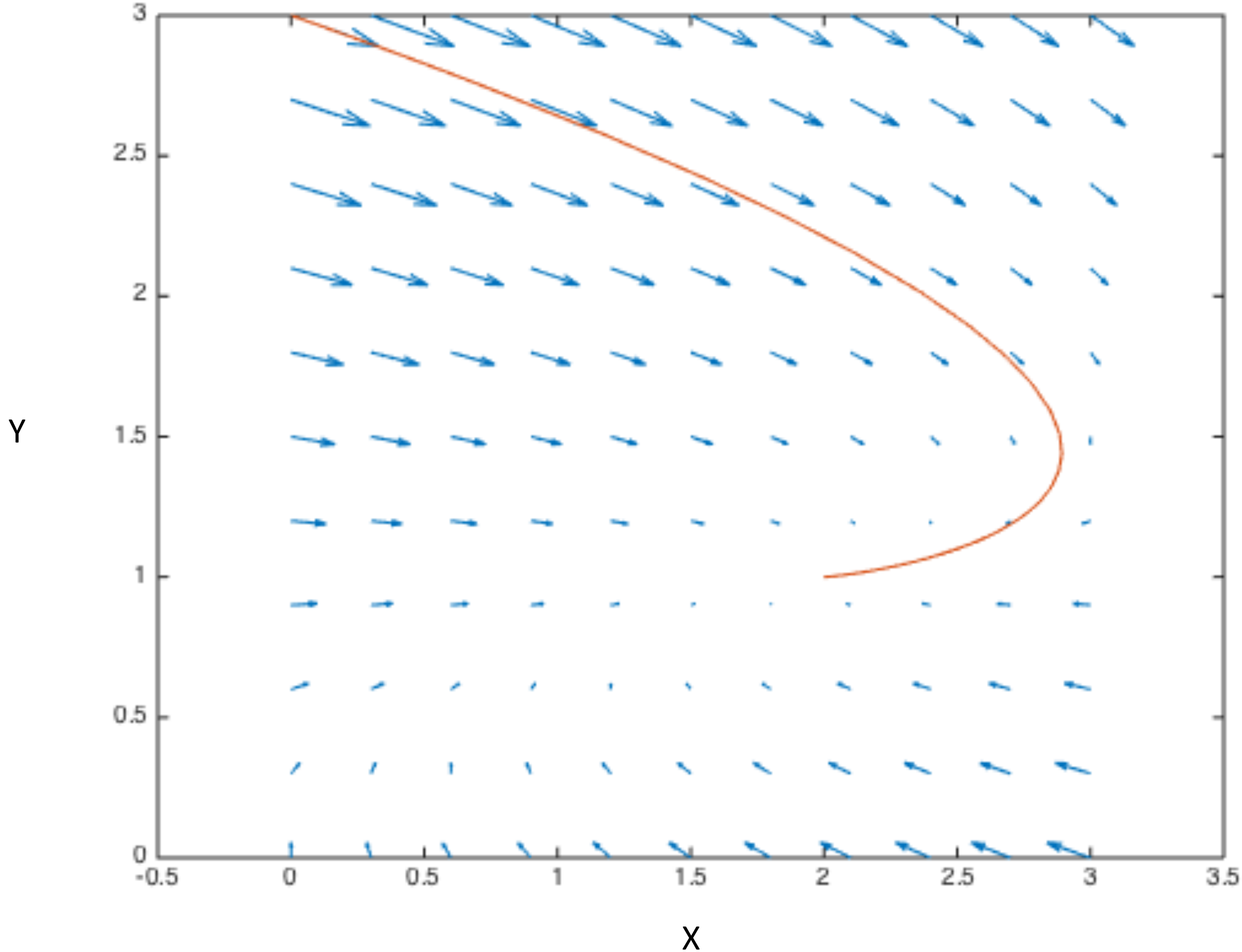
X=[0:3], Y=[0:3]



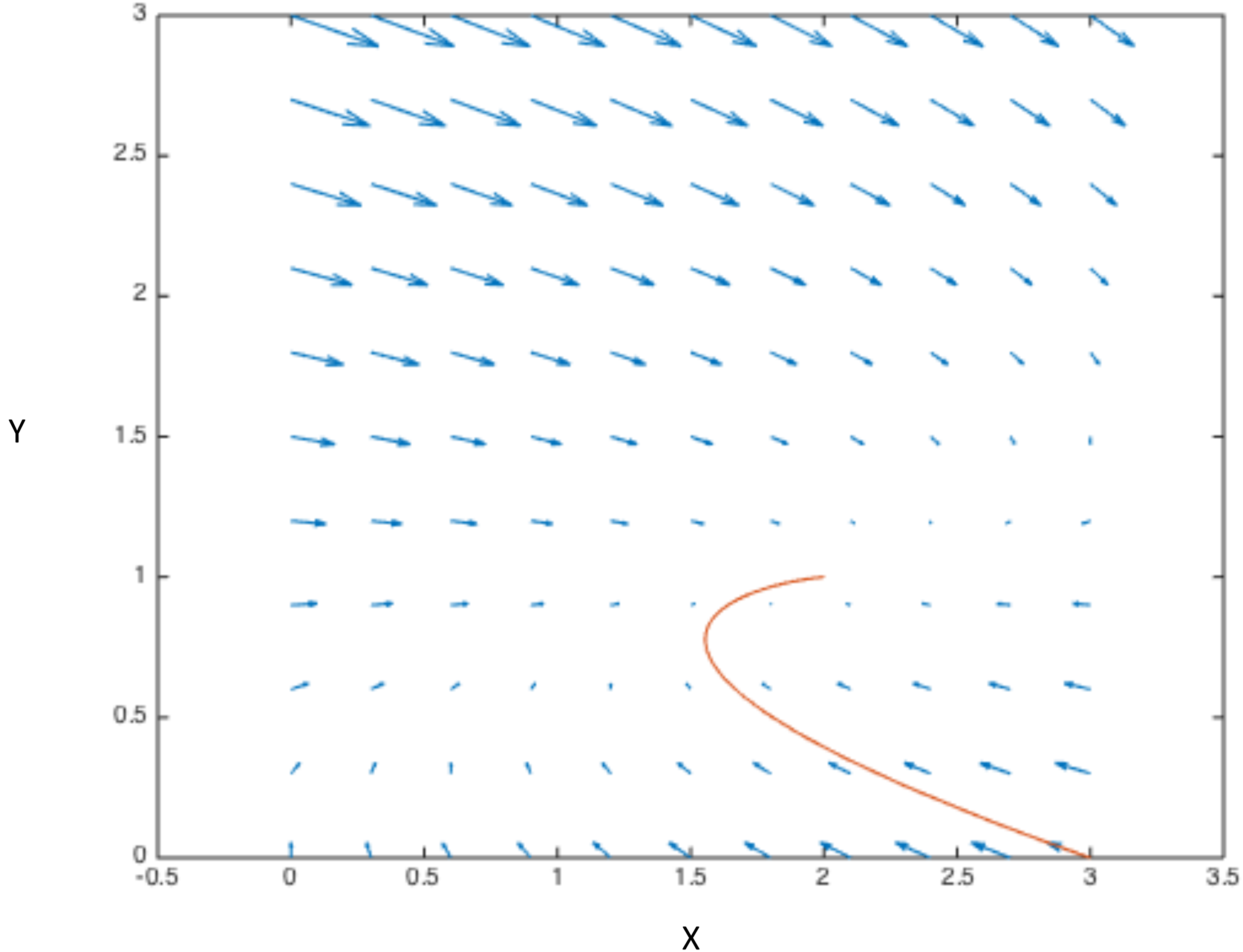
Y
X
3
2.5
2
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
3.5



Y
X
3
2.5
2
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
3.5



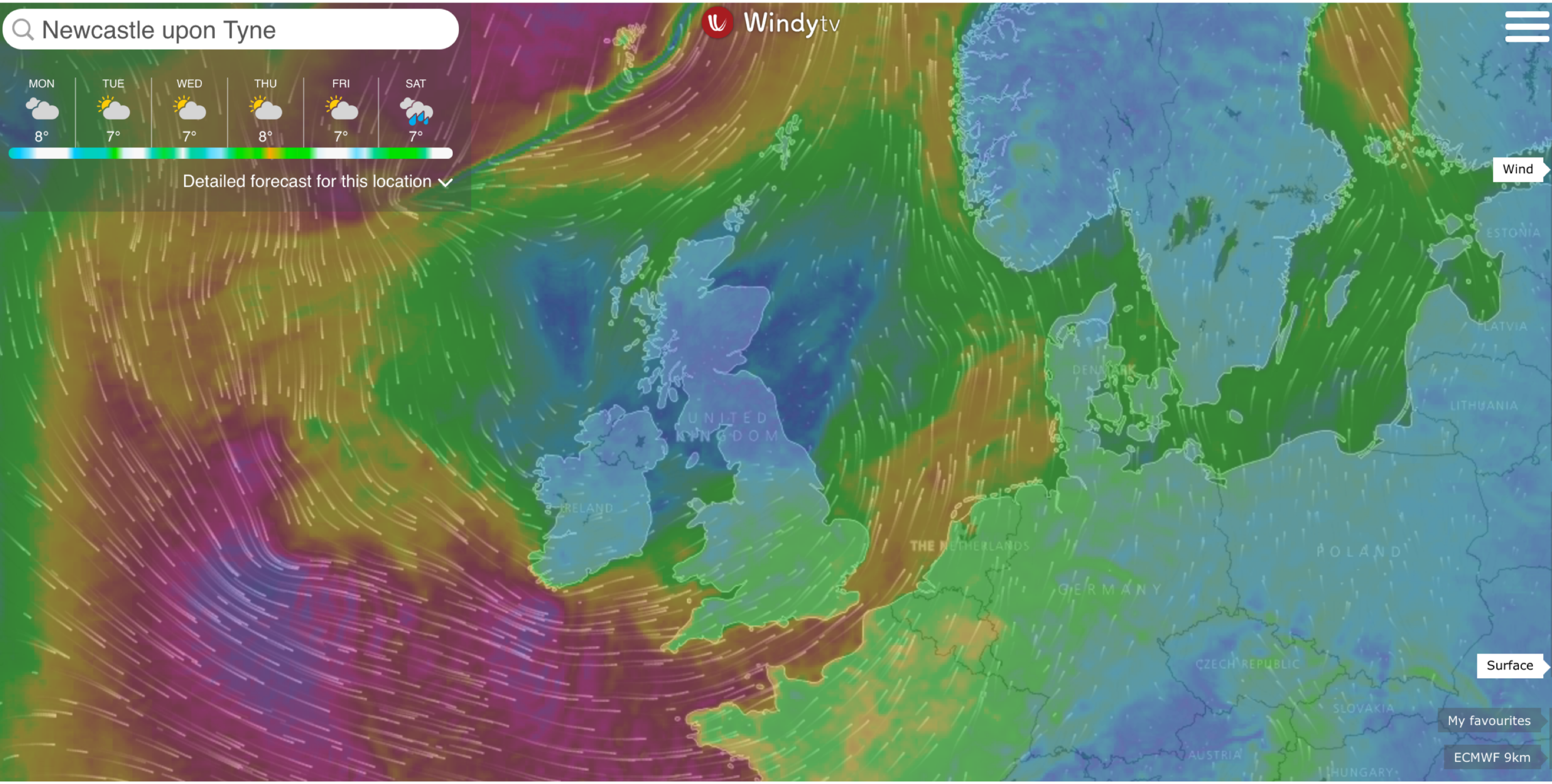
3
2.5
2
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
Y
X



3
2.5
2
1.5
1
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
Y
X

Example of vector field and trajectory

https://www.windytv.com/?54.988,-1.619,5



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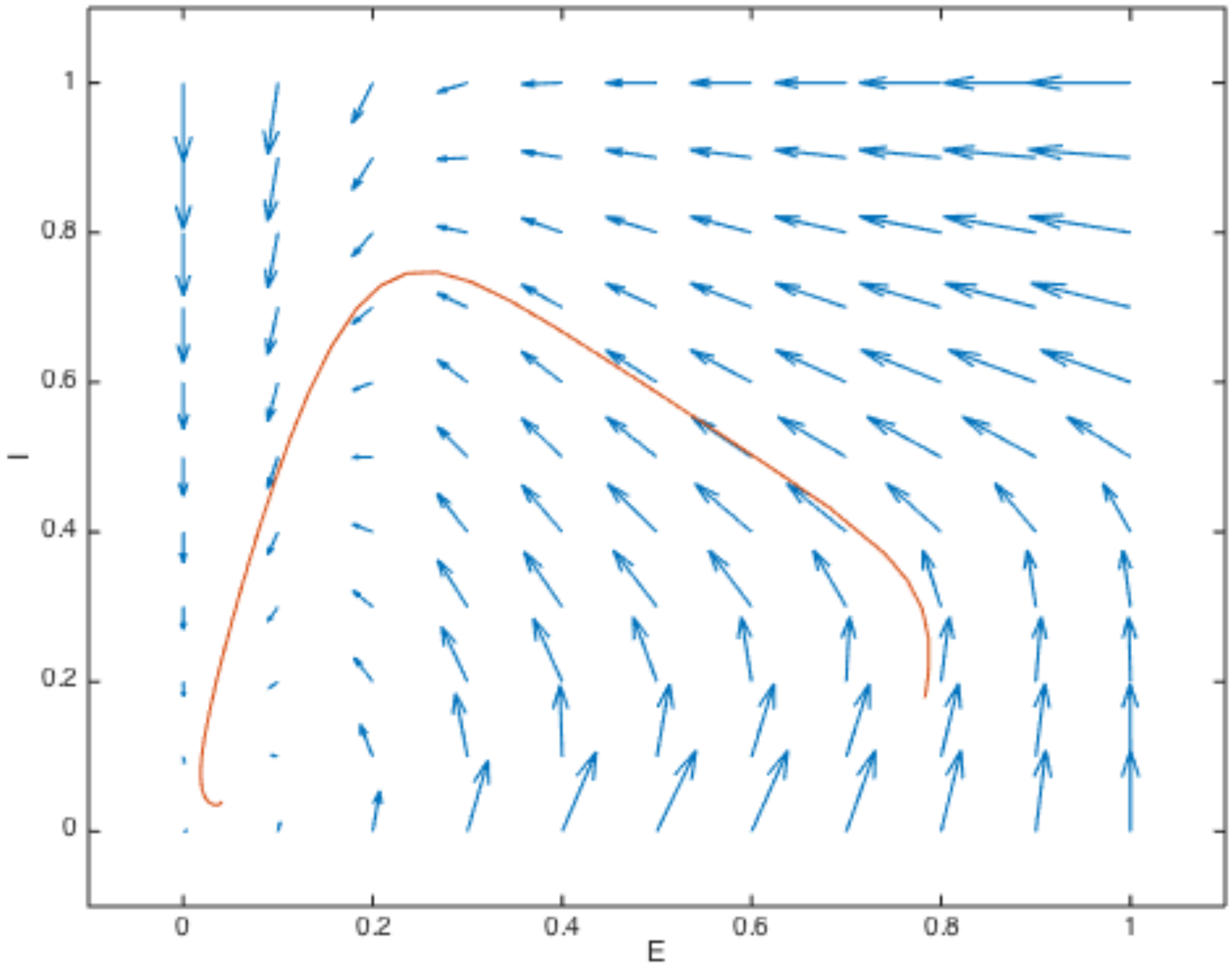
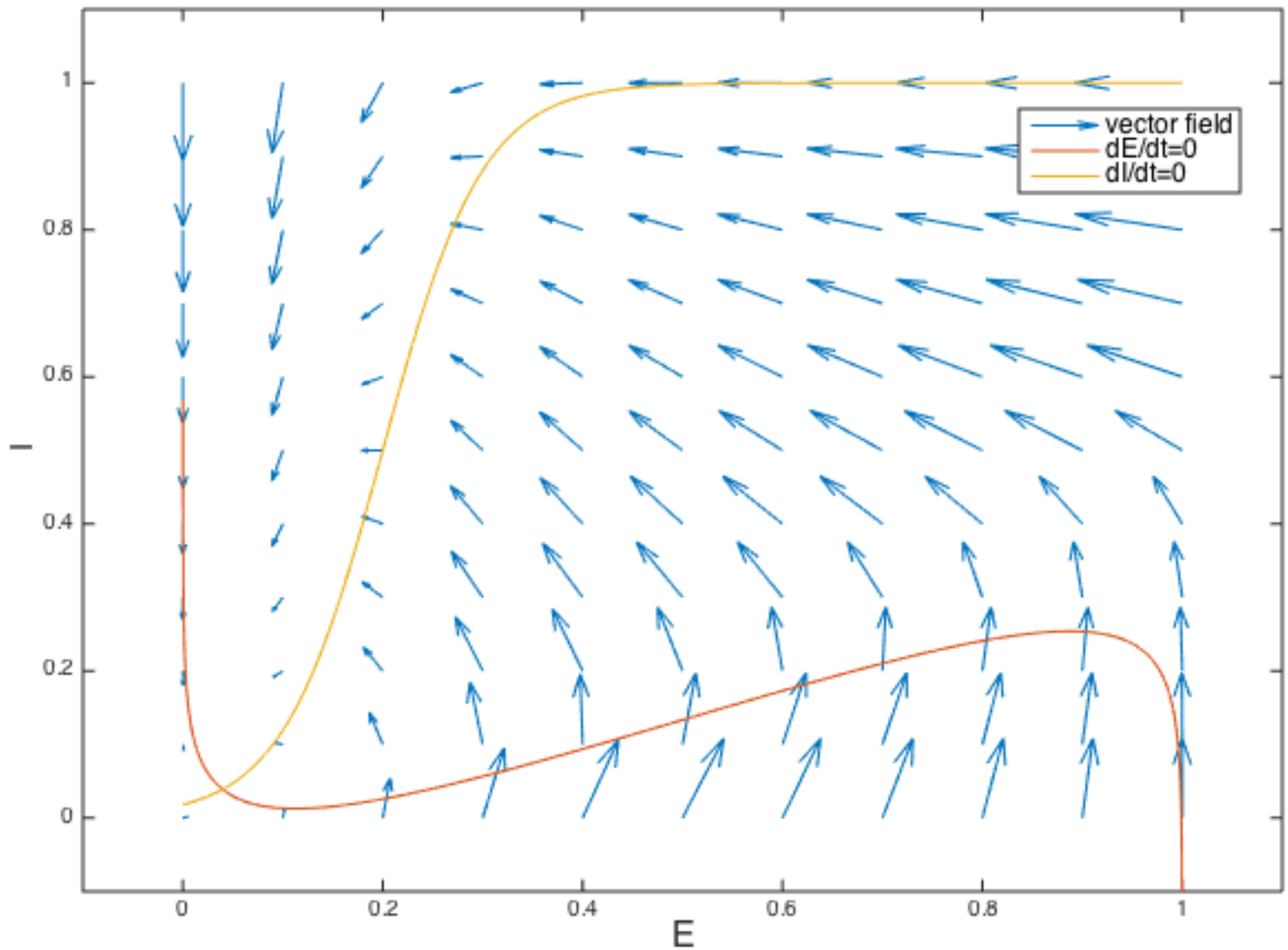
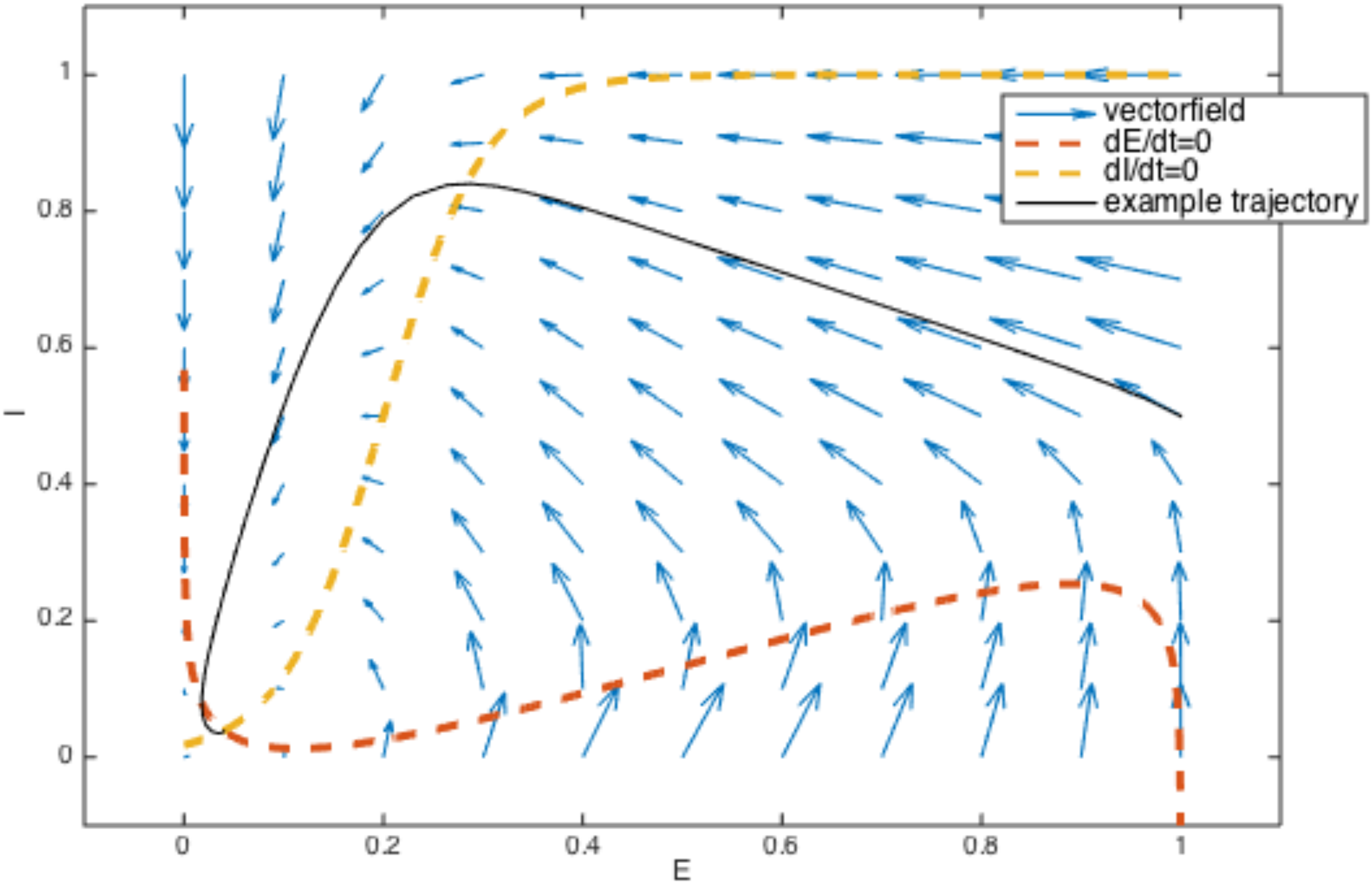
Phase space with vector field

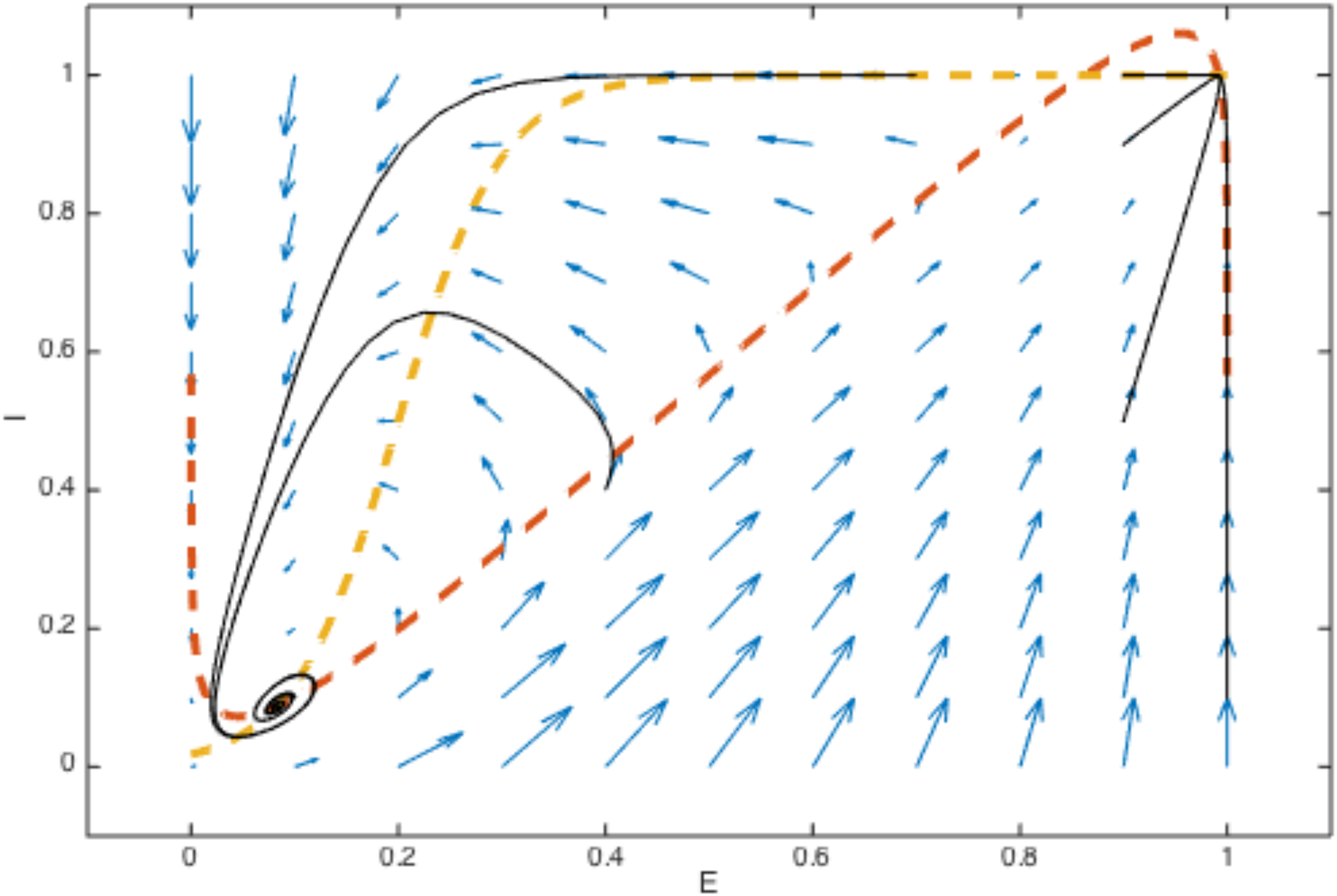


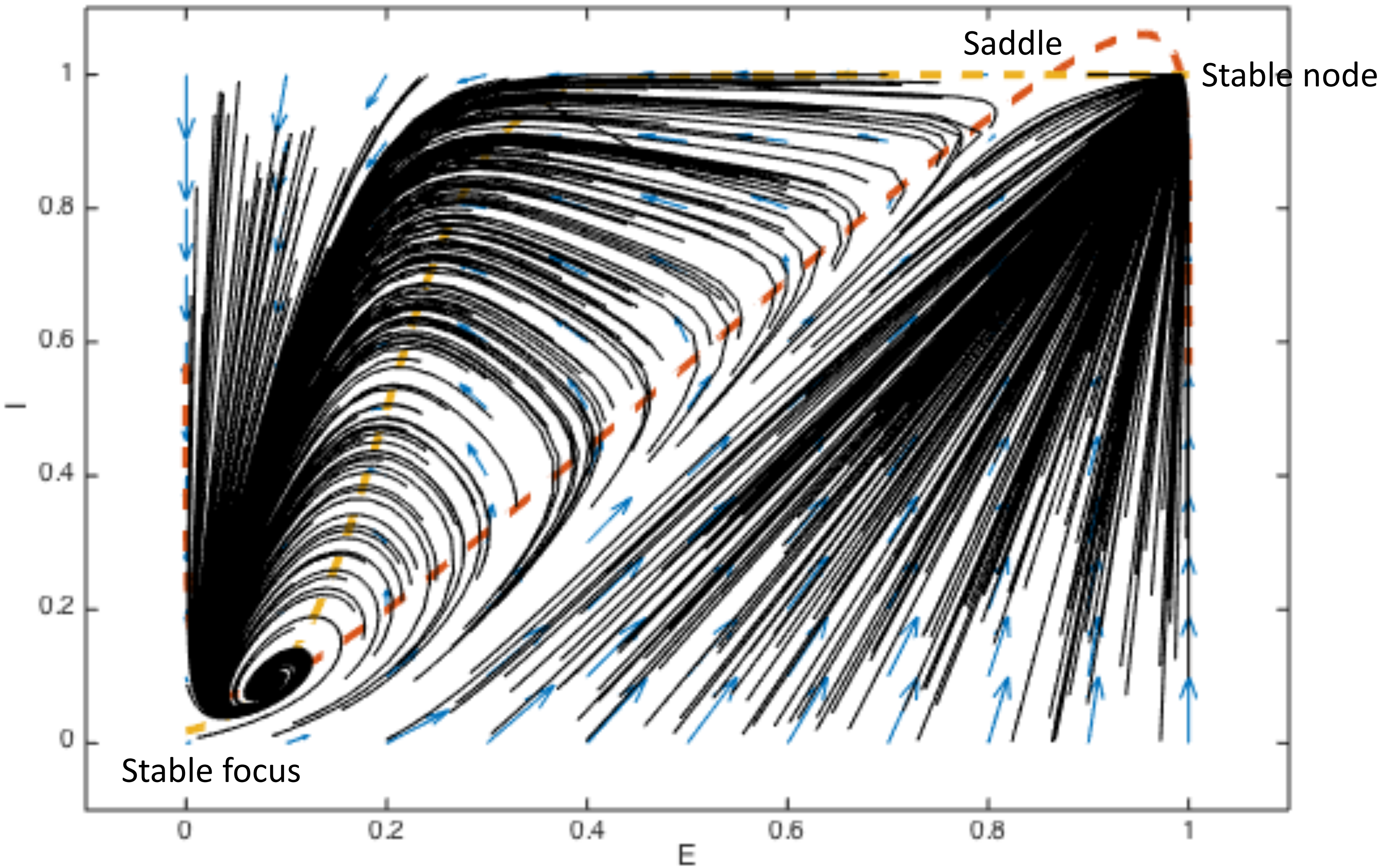
Phase space with vector field and nullcline



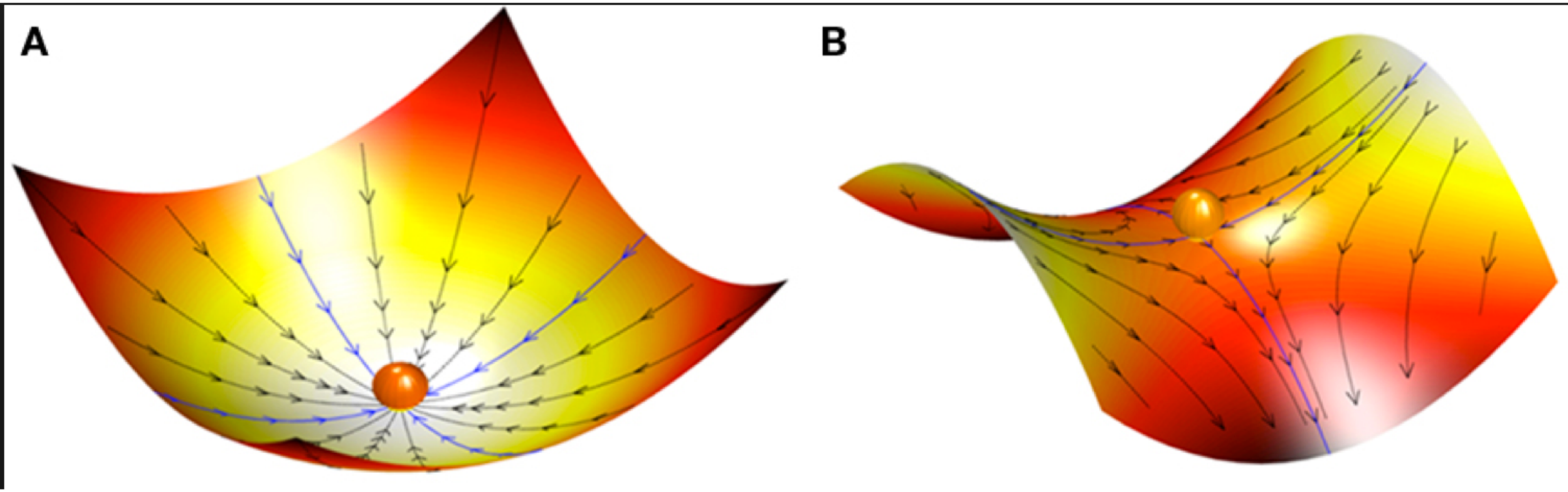

vectorfield
dE/dt=0
dI/dt=0
example trajectory
I
E
0
0.2
0.4
0.6
0.8
1

Increase self excitation (parameter a)

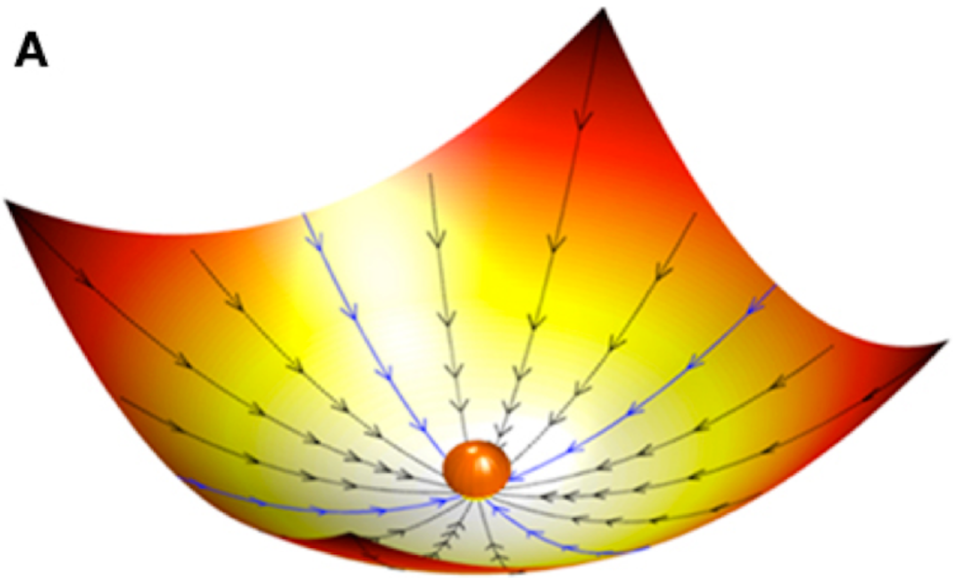


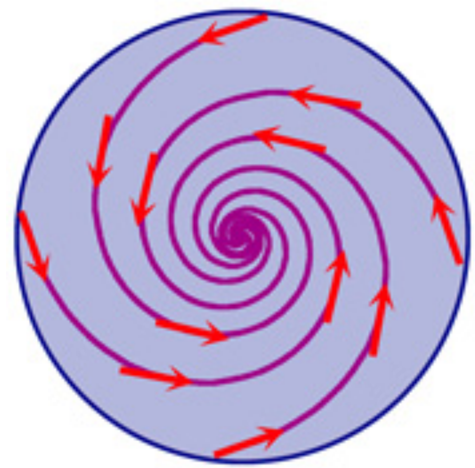

Saddle
Stable node
Stable focus
1
0.8
0.6
0.4
0.2
0
I
0
0.2
0.4
0.6
0.8
1
E

Illustrating fixed point stability

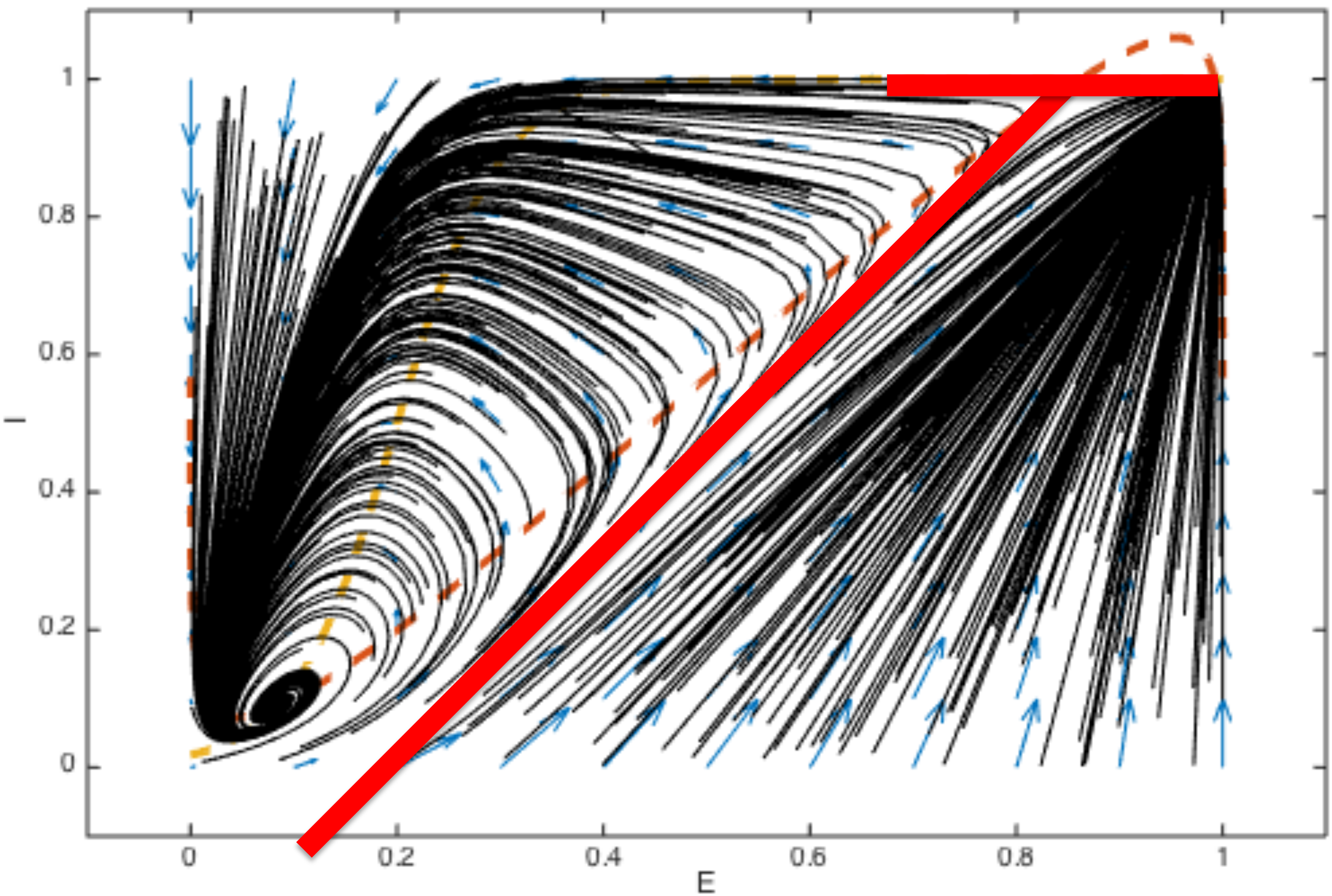


Node vs. Focus

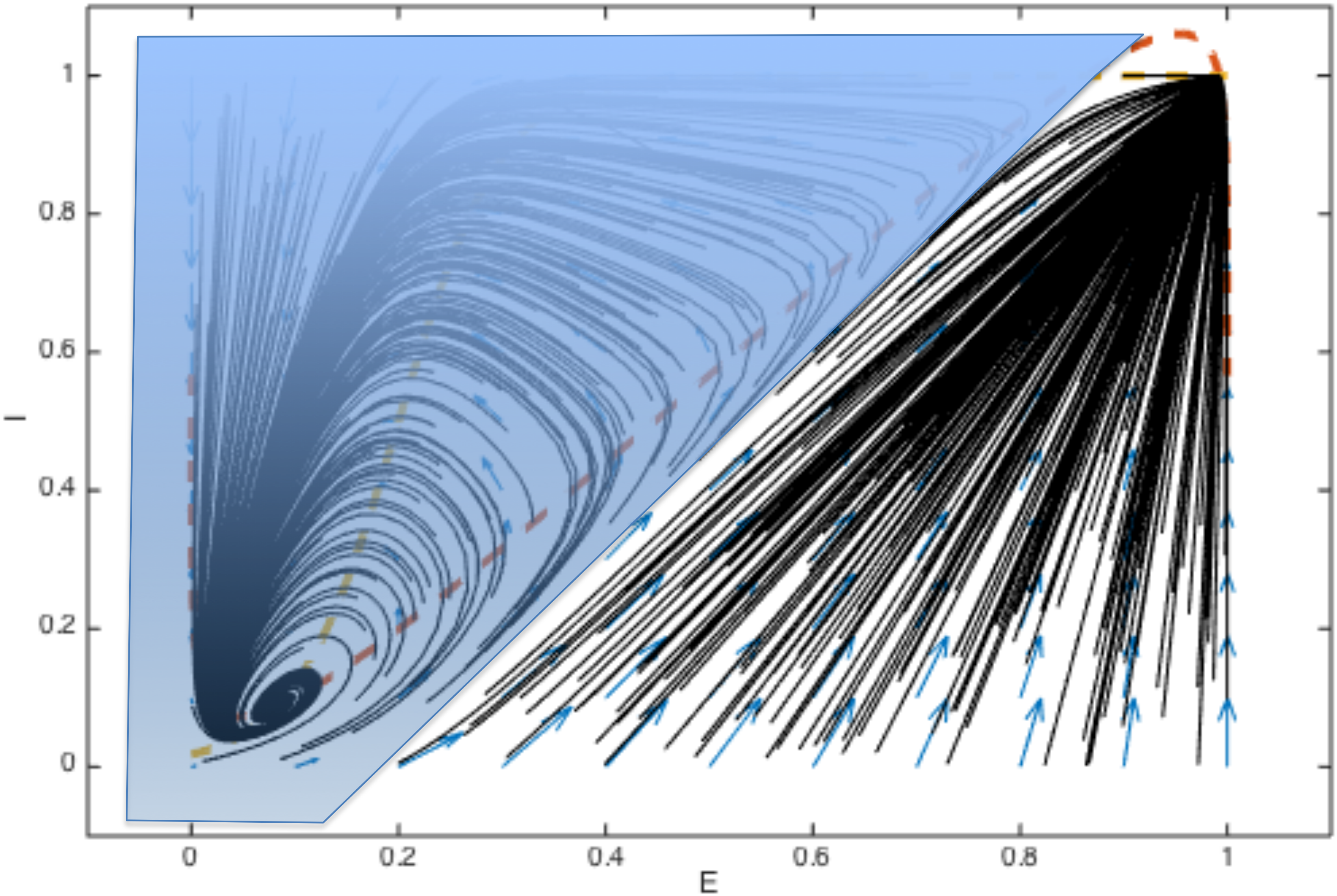




Saddle and separatrix



Basin of attraction



Terminology recap

- Phase space/state space
- Vectorfield
- Fixed point (stable/unstable, focus/node)
- Nullcline
- Saddles, separatrix
- Bistability

Solving for a steady state analytically (for yesterday's exercises)

$$dx/dt = k*x + c$$

$$dx/dt = 0$$

$$k*x_{ss} + c = 0$$

$$k*x_{ss} = -c$$

$$x_{ss} = -c/k$$

Plotting phase space, vectorfields, nullclines

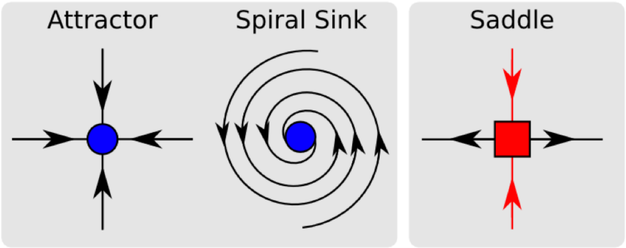
Google: pplane matlab

http://math.rice.edu/~dfield/#8.0

Summary

Flow: The rate of change of the system at every state. On a phase portrait, arrows represent the flow. They have a length and direction proportional to the size and direction of change.

Steady states: States of the system for which the rate of change is zero. A steady state is **stable** if, upon perturbation, a trajectory will reconverge to it. Otherwise, the steady state is **unstable.**



Stable steady states

Unstable steady state

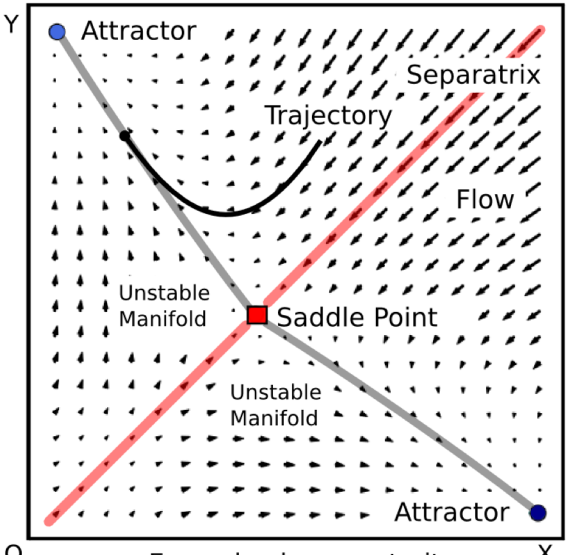
Trajectory: Curve that describes the change of state variables over time.

Basin of attraction: Region around a stable state from which all trajectories converge towards it.

Separatrix: Curve that separates basins of attraction.

Unstable manifold: Trajectory joining a saddle to an attractor.

A phase portrait represents a dynamical system in a graphical way. The approach has the advantage that trajectories can be understood in terms of their drivers, i.e. the distinctive features of a system.



Example phase portrait

Quiz

- Where do nullclines come from?
- What do nullclines indicate in phase space?
- What is the intersection of nullclines?

- Where does the vector field come from?
- What is its relationship to nullclines?
- Why is the vector field useful?

Quiz

- What effects can parameters have on the vector field and nullclines?
- What is a fixed point?
- Trick question: Does the fixed point depend on initial conditions?
- Trick question: Why is the vectorfield on a grid system?
- Why do trajectories never intersect?