Overview

• What are dynamical systems?
• How to interpret a differential equation
• How to analyse differential equation systems
• How to solve differential equation systems
• Stability analysis, multistability
• Oscillatory solutions
• Parameter variations, bifurcations
• Choice of cool stuff: Chaos, turbulence, spatio-temporal systems, slow-fast systems, transients, and more.
Back to the example: Feedback inhibition

[X] -> Protein X concentration
[Y] -> Protein Y is the dimer of XX

\[
\begin{align*}
\frac{d[X]}{dt} &= -kX - rX^2 + (g-Y) \\
\frac{d[Y]}{dt} &= rX^2 - lY
\end{align*}
\]

k=0.1: degradation rate of X
l=0.8: degradation rate of Y
r=0.5: dimerisation rate
g=2: production rate of X
Questions about the system

• What is the long-term dynamics of the system? (i.e. ultimately, what levels of X and Y do I get?)

• How does this change with different parameters?
Determine the fixed point

dX/dt=0 and dY/dt=0

... 

X=1.3705, Y=1/1739
Phase space
Stability of the fixed point

- Determine if the rate of successive iterations approach or leave the fixed point
- To do this we need to look at the change in the vector field near the fixed point
- To do this we find apply Taylor expansion to the vector field and use the first order approximation (Jacobian).
- This is also called an linear stability analysis
Consider the general system of two first-order ordinary differential equations

\[
\begin{align*}
\dot{x} &= f(x, y) \\
\dot{y} &= g(x, y).
\end{align*}
\]

Let \(x_0\) and \(y_0\) denote fixed points with \(\dot{x} = \dot{y} = 0\), so

\[
\begin{align*}
f(x_0, y_0) &= 0 \\
g(x_0, y_0) &= 0.
\end{align*}
\]

Then expand about \((x_0, y_0)\) so

\[
\begin{align*}
\delta \dot{x} &= f_x(x_0, y_0) \delta x + f_y(x_0, y_0) \delta y + f_{xy}(x_0, y_0) \delta x \delta y + \ldots \\
\delta \dot{y} &= g_x(x_0, y_0) \delta x + g_y(x_0, y_0) \delta y + g_{xy}(x_0, y_0) \delta x \delta y + \ldots.
\end{align*}
\]

To first-order, this gives

\[
\frac{d}{dt} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix},
\]

where the \(2 \times 2\) matrix is called the stability matrix.
For the feedback inhibition:

\[
J = \begin{bmatrix}
-k-r^*x & -1 \\
rx & -1
\end{bmatrix}
\]

\[
J = [-0.1-0.5*1.3705 & -1; 0.5*1.3705 & -0.8]
\]

\[
\text{eig}(J)
\]

\[
\text{ans} = 
\begin{bmatrix}
-0.7926 + 0.8278i \\
-0.7926 - 0.8278i
\end{bmatrix}
\]
Interpreting eigenvalues

• Eigenvalues tell you about the speed of exponential decay or rise to/from a fixed point.

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>Effect on system when disturbed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive real number</td>
<td>Driven away from steady-state value</td>
</tr>
<tr>
<td>Negative real number</td>
<td>Driven back to steady-state value</td>
</tr>
<tr>
<td>0</td>
<td>Remains at position to which it was disturbed</td>
</tr>
<tr>
<td>Identical to another eigenvalue</td>
<td>Effects can not be determined</td>
</tr>
<tr>
<td>Complex, positive real number</td>
<td>Oscillates around steady-state value with increasing amplitude</td>
</tr>
<tr>
<td>Complex, negative real number</td>
<td>Oscillates around steady-state value with decreasing amplitude</td>
</tr>
<tr>
<td>Imaginary</td>
<td>Oscillates around steady-state value with constant amplitude</td>
</tr>
</tbody>
</table>
Marble Analogy

**Case I: stable**

![Graph showing a stable equilibrium point](image1.png)

Small perturbations left or right will cause the marble to decay back to the steady state position.

→ Negative real eigenvalue

**Case II: unstable**

![Graph showing an unstable equilibrium point](image2.png)

Small perturbations left or right will cause the marble to decay away from the steady state position ($x_{ss}$).

→ Positive real eigenvalue

**Case III: Saddle point**

![Graph showing a saddle point](image3.png)

Small perturbations in $y$ are stable, while perturbations in $x$ are unstable (saddle point), thus overall point is unstable!

→ Positive and negative real eigenvalues
What about not fixed points?

• 1. Look at the vector field!!!
• 2. Can measure how quickly two trajectories diverge in phase space that started from very similar initial conditions.
• This measure is called the Lyapunov exponent.
• In most cases in real life this exponent is derived numerically. (Can also be found for experimental data!)
• A big Lyapunov exponent often indicates Chaos
Terminology recap

- Focus, node
- Stable, unstable, saddle
- Jacobian
- Eigenvalues
- Linear stability analysis
- Lyapunov exponent