Terminology recap

- Variable or state
- Differential equation
- Initial condition
- Trajectory
- Parameter
- Steady state/fixed point/equilibrium

Terminology recap

- Phase space/state space
- Vectorfield
- Fixed point (stable/unstable, focus/node)
- Nullcline
- Saddles, separatrix
- Bistability

Terminology recap

- Order of ODE solver
- Step size
- Variable/adaptable stepsize solvers
- Eulers method
- Heun’s method
- Runge Kutta method

Dynamical Systems

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Lecture 4 of 5

Overview

- What are dynamical systems?
- How to interpret a differential equation
- How to analyse differential equation systems
- How to solve differential equation systems
- Stability analysis, multistability
- Oscillatory solutions
- Parameter variations, bifurcations
- Choice of cool stuff: Chaos, turbulence, spatio-temporal systems, slow-fast systems, transients, and more.

Back to: Neural population model

E -> fractional firing of excitatory neural population

I -> fractional firing of inhibitory neural population

$$dE/dt = -E + S(a*E - b*I + P)$$

$$dI/dt = -I + S(c*E - d*I + Q)$$

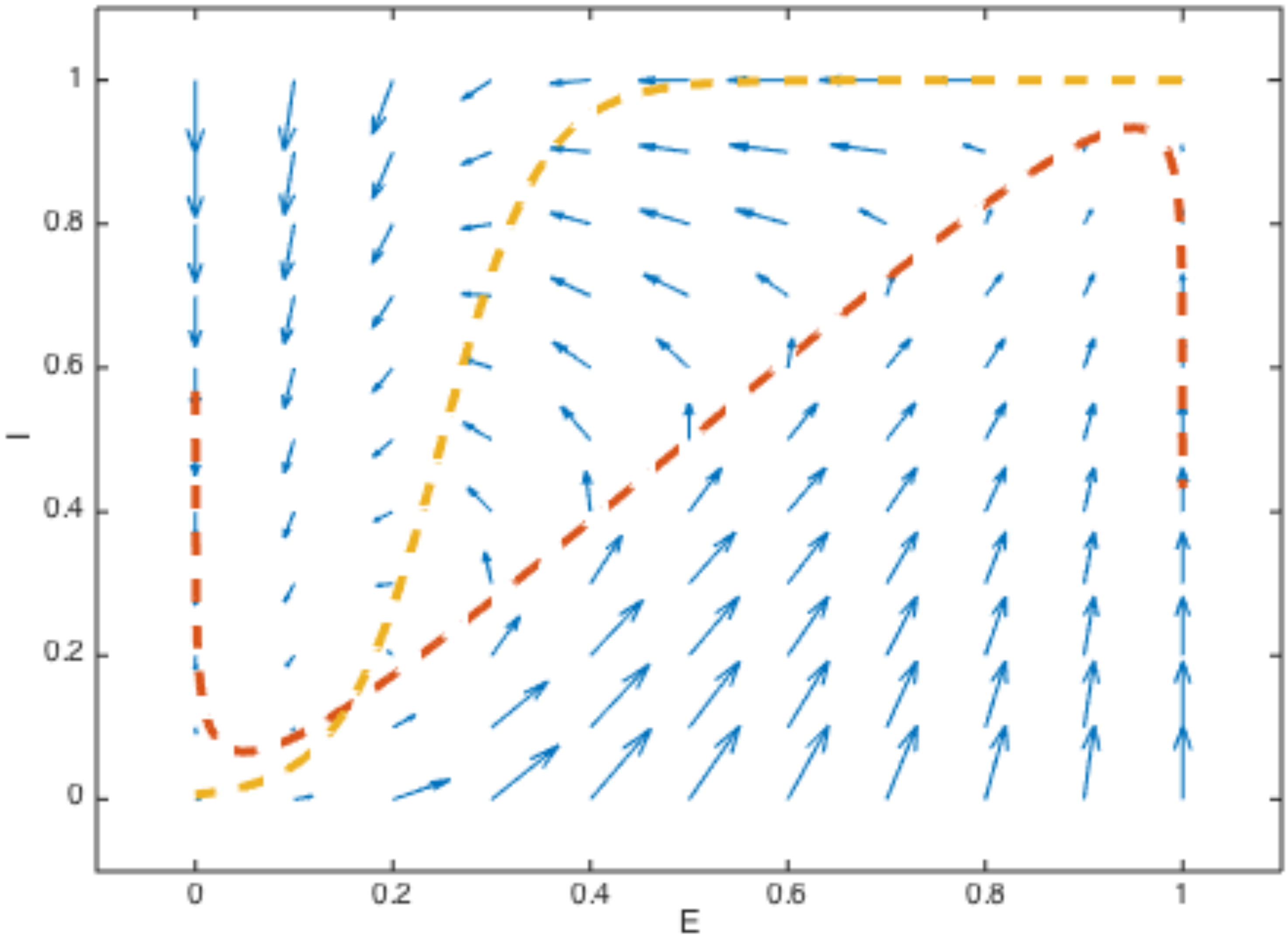
Also known as Wilson-Cowan equations

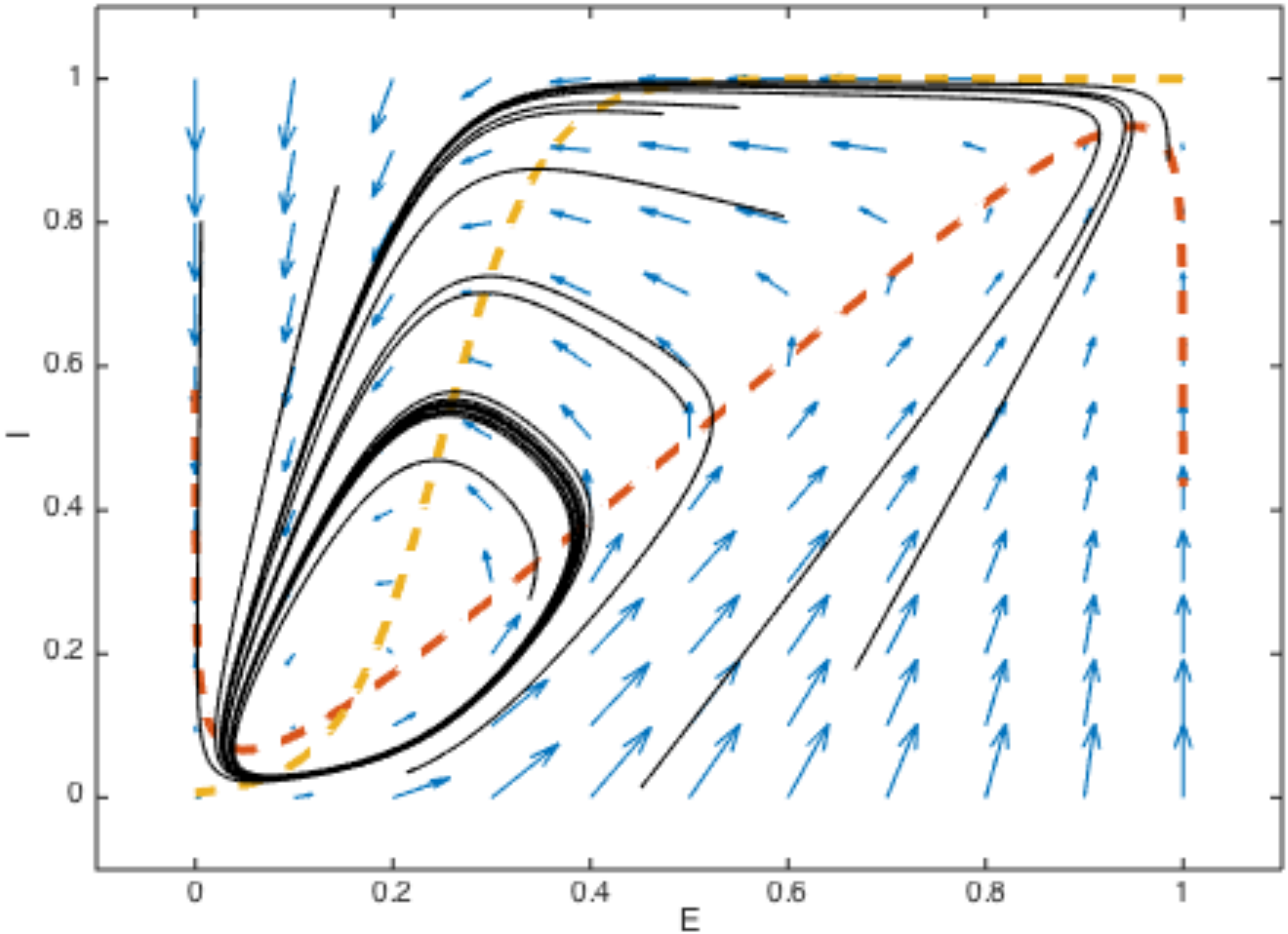
a,b,c,d: connectivity weights

P,Q: baseline input to the populations

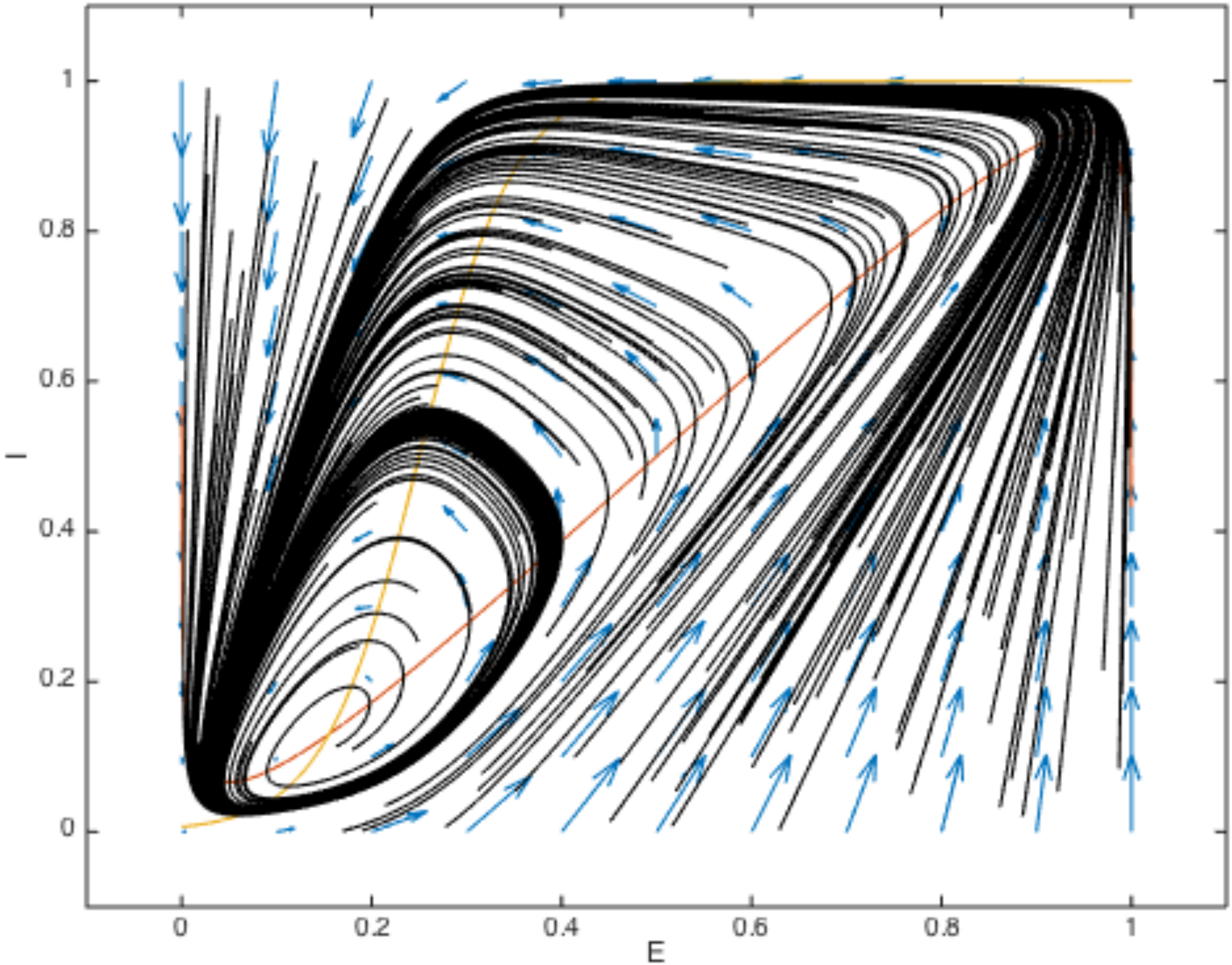
S: sigmoid function

P=1, Q=-1, a=21



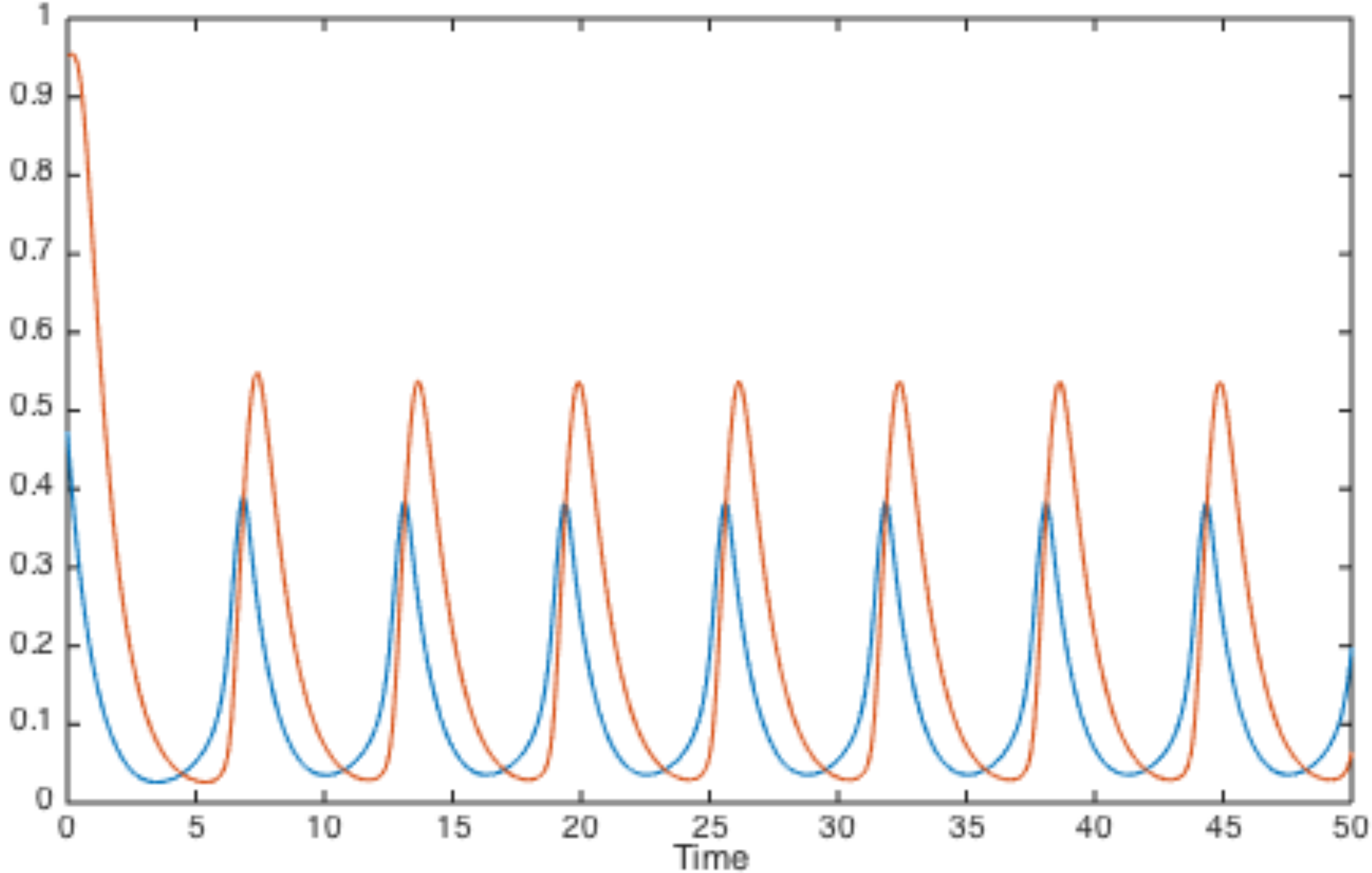


1
0.8
0.6
0.4
0.2
0
I
0
0.2
0.4
0.6
0.8
1
E



1
0.8
0.6
0.4
0.2
0
I
0
0.2
0.4
0.6
0.8
1
E

Time series of an example trajectory



A note on Limit cycles

- Not as neat to deal with as fixed points

(no $dx/dt=0$)

- In general no analytic expression possible
- Difficult to obtain the stability (yes, unstable, stable and saddle limit cycles all exist!)

Overview

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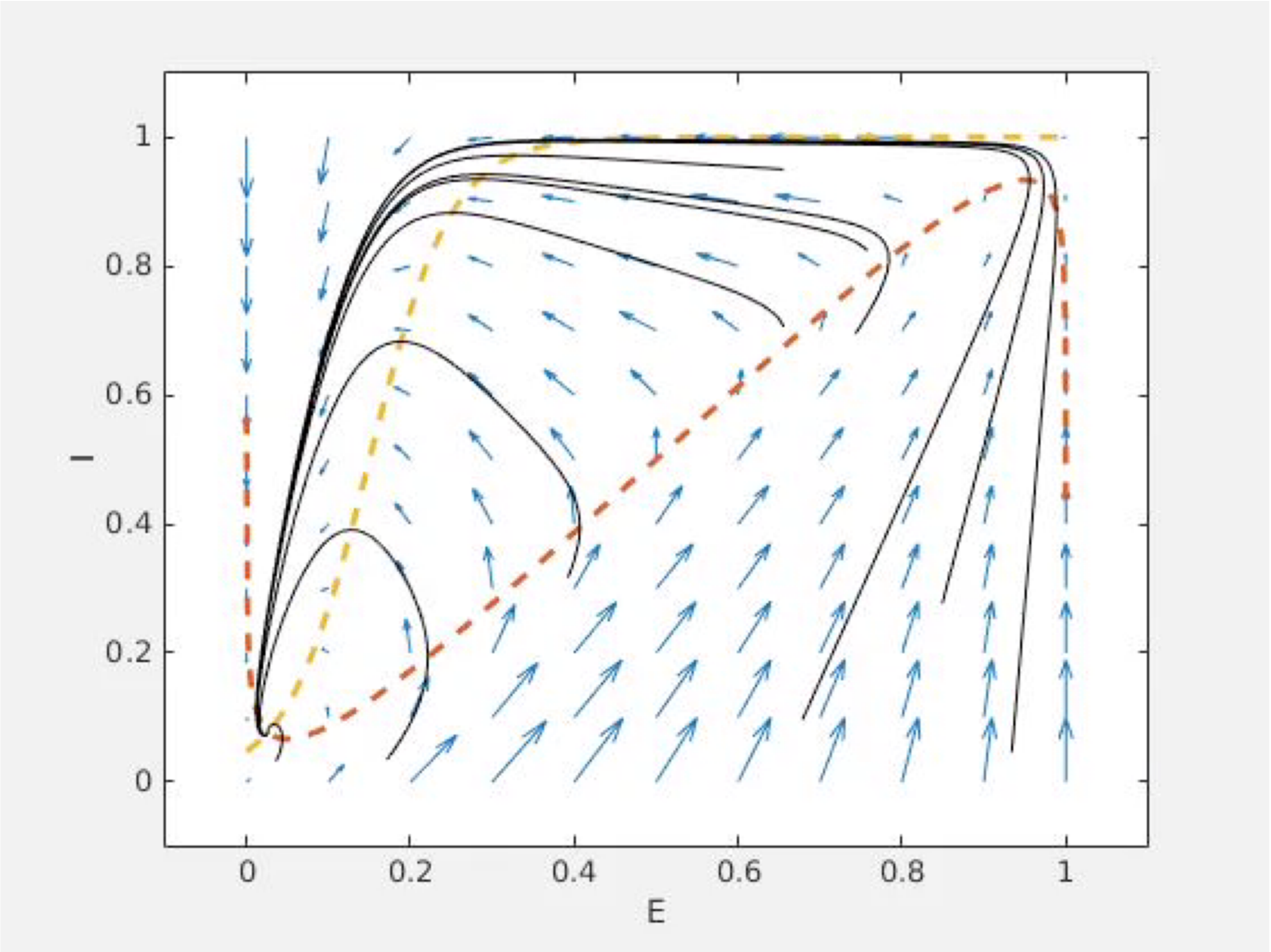
How does the phase space change with parameters? Changing Q



Critical state transitions in nature

- Small change in parameter(s) that lead to a sudden change in the qualitative behaviour of a system
- Epidemics
- Trends in society
- Gas -> liquid -> solid
- Superconductivity

Types of state transitions in dynamical systems (bifurcation theory)

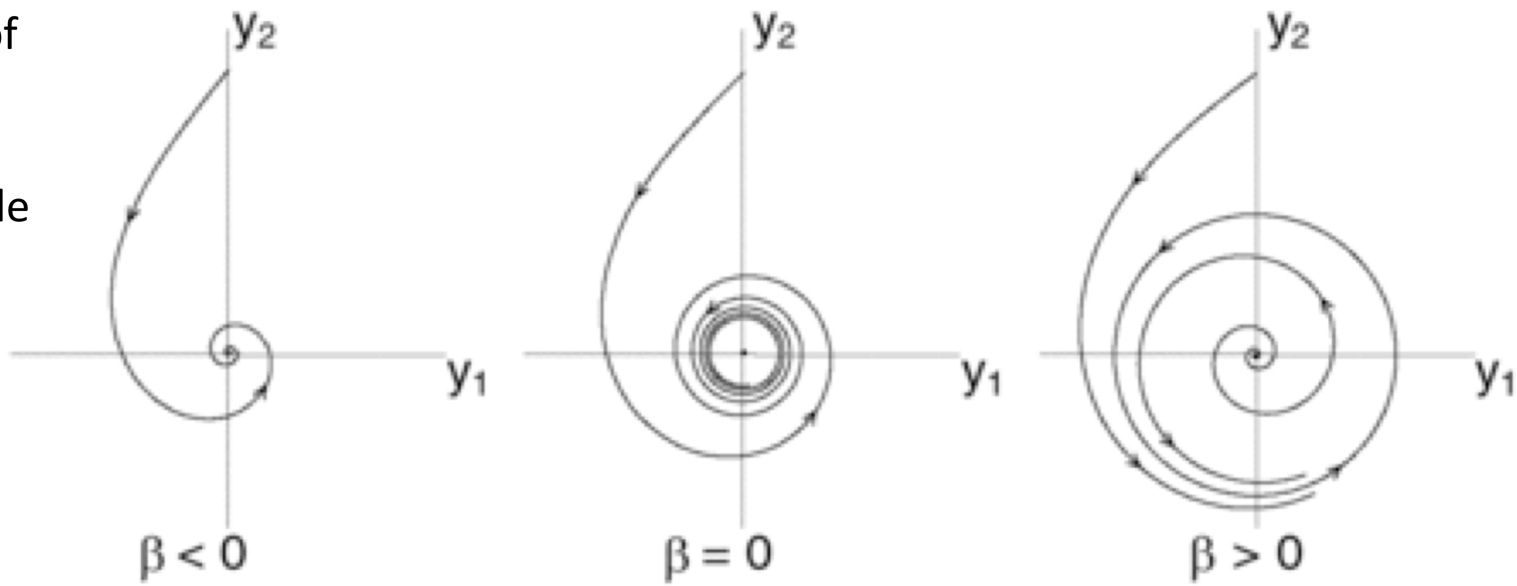
- Hopf
- Saddle-Node
- Homoclinic
- …

Andronov-Hopf bifurcation

- Supercritical Hopf: a stable focus becomes unstable at the bifurcation point, and a stable limit cycle arises.
- Subcritical Hopf: an unstable focus becomes stable at the bifurcation point, and an unstable limit cycle arises.
- Eigenvalues cross the imaginary axis at bifurcation point.

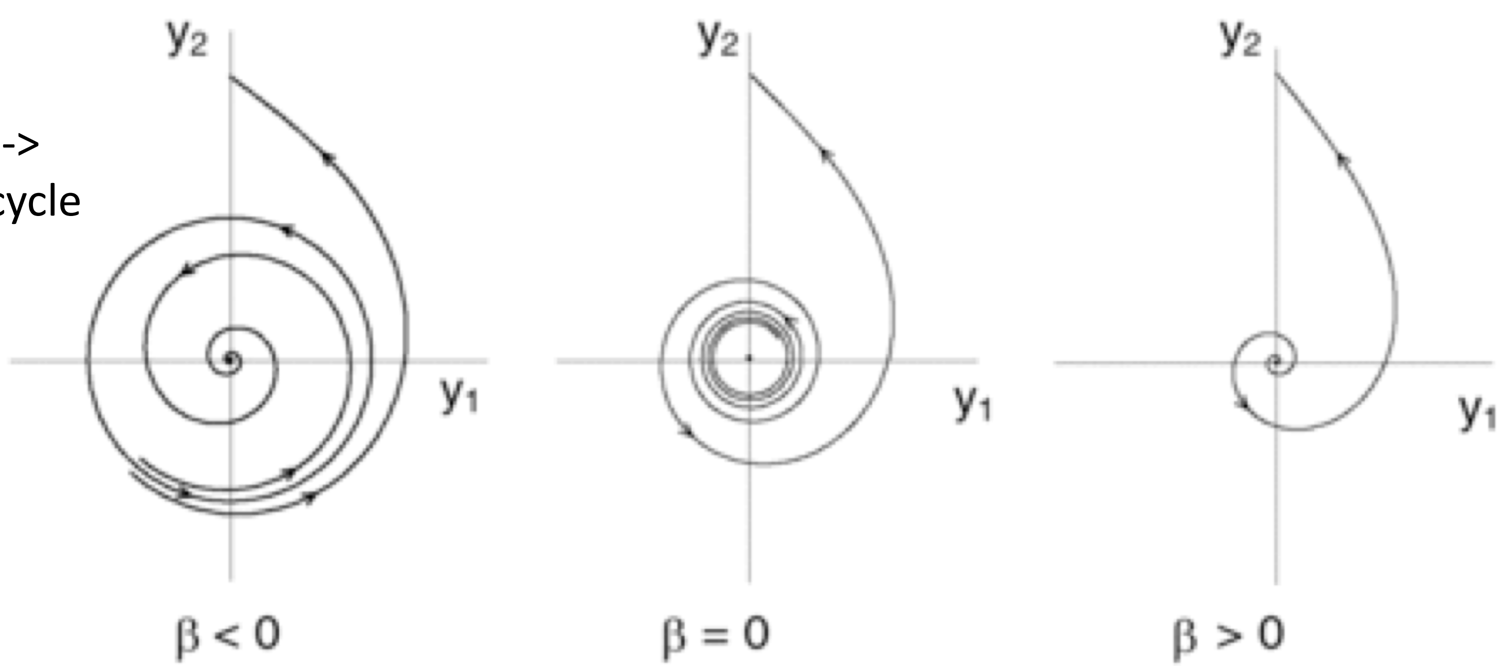
Supercritical Hopf

Stable focus ->
Stable limit cycle

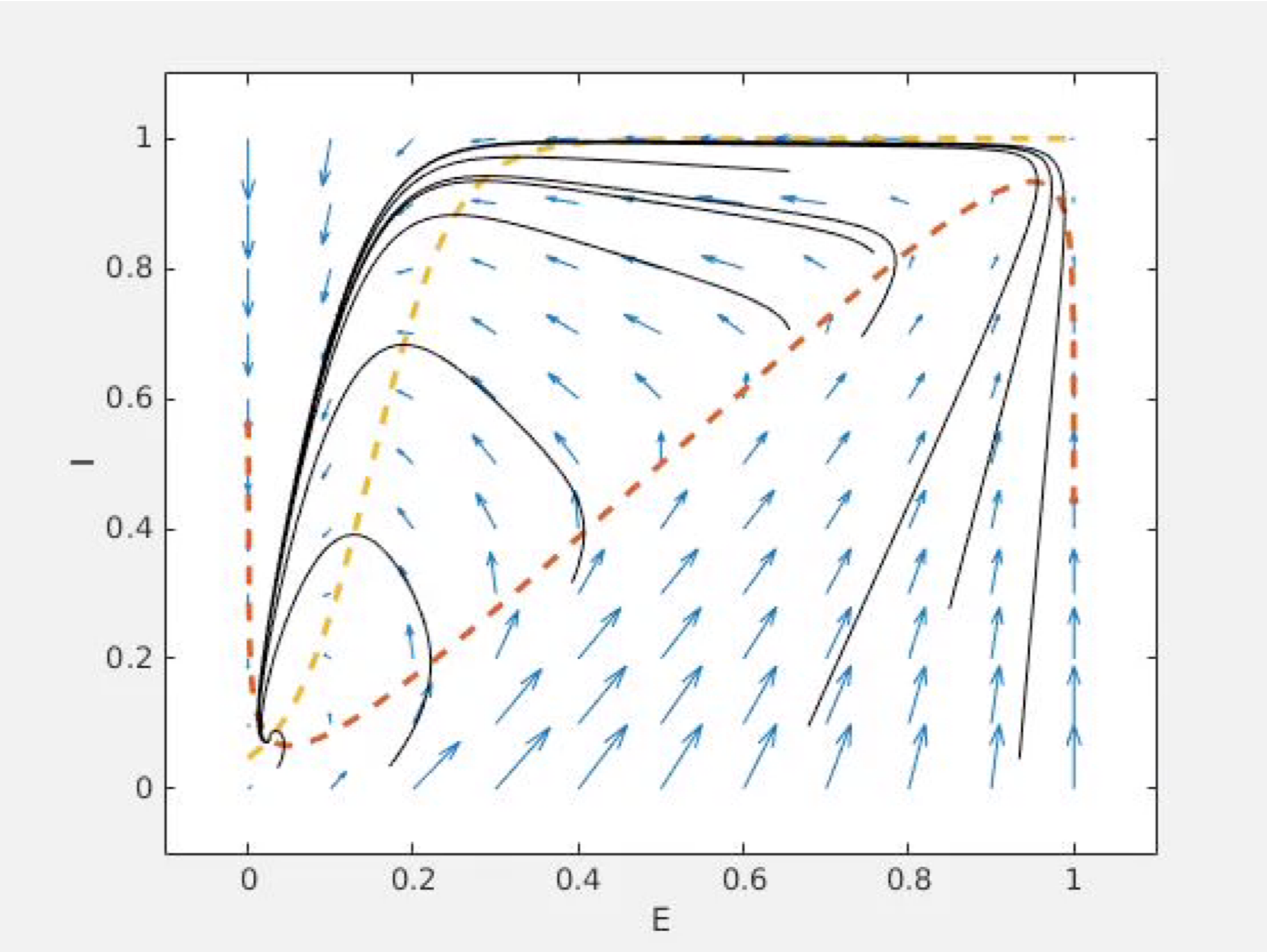


Subcritical Hopf

Unstable focus ->
Unstable limit cycle

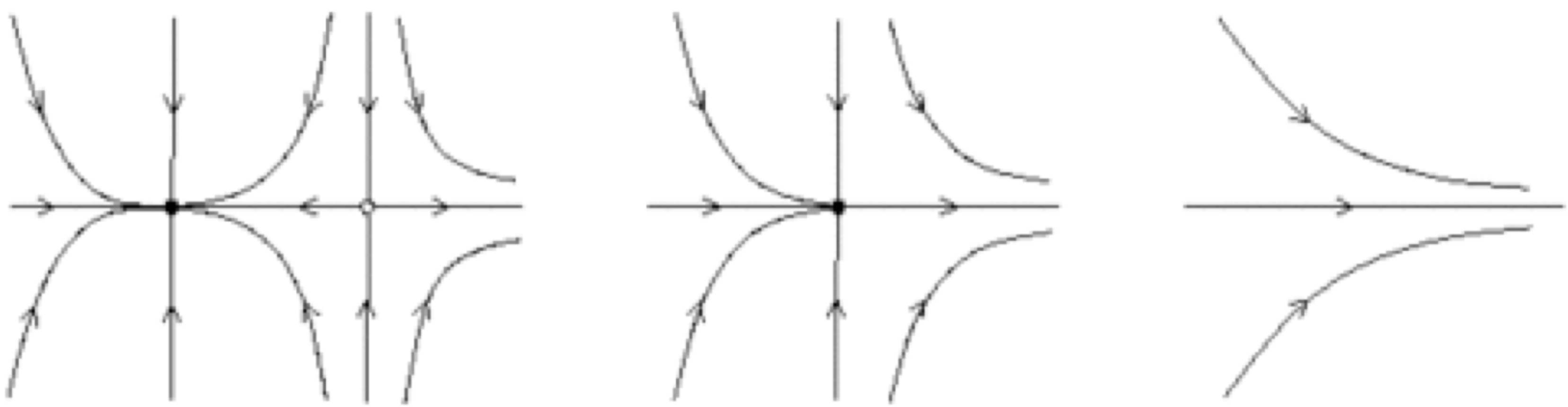


Changing Q: two Hopf bifurcations!



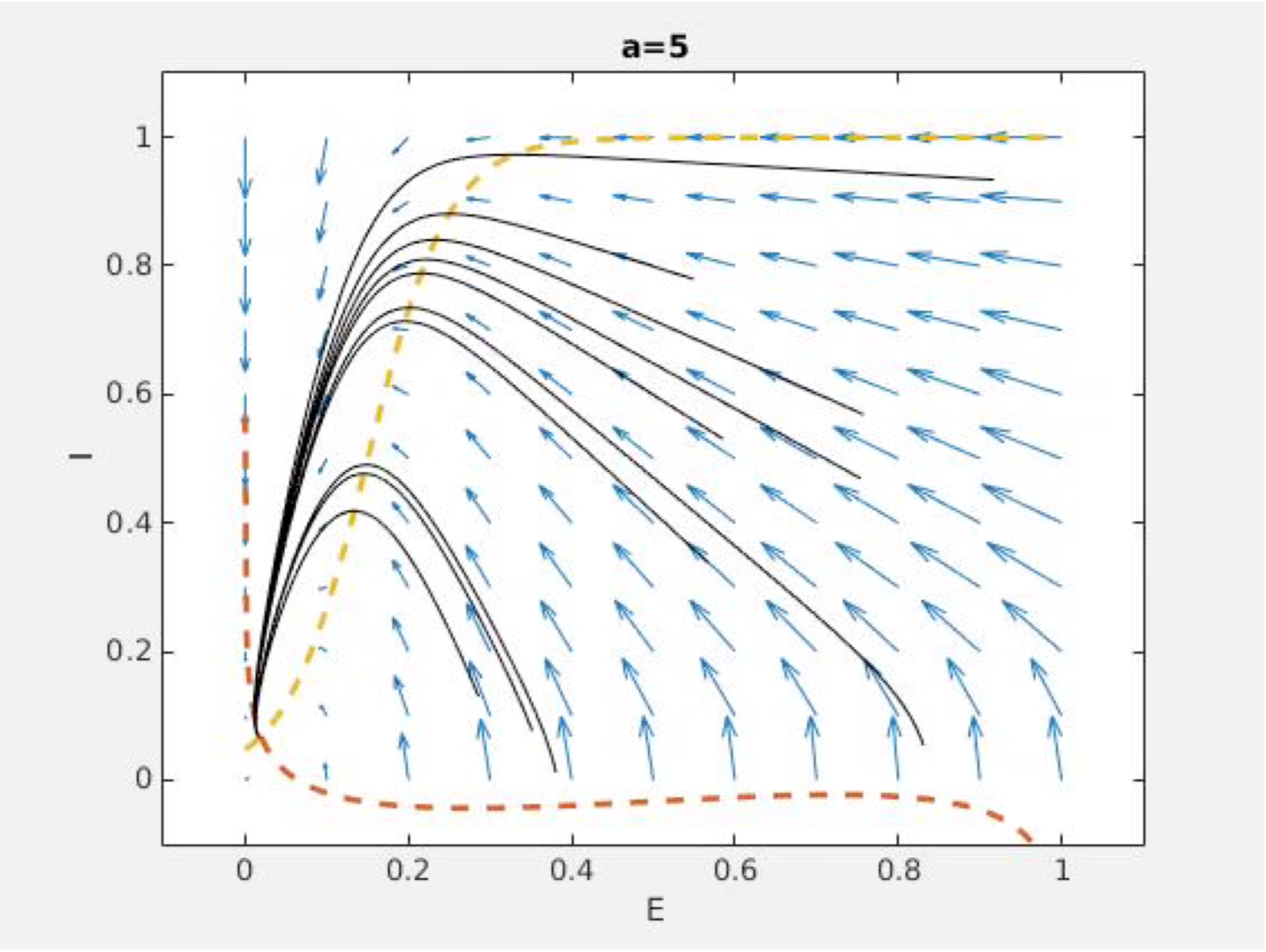
Saddle-Node bifurcation

- Also called fold, or limit point bifurcation
- Two fixed points collide and disappear (or two fixed point are born)
- Can often be understood well using nullclines in phase space



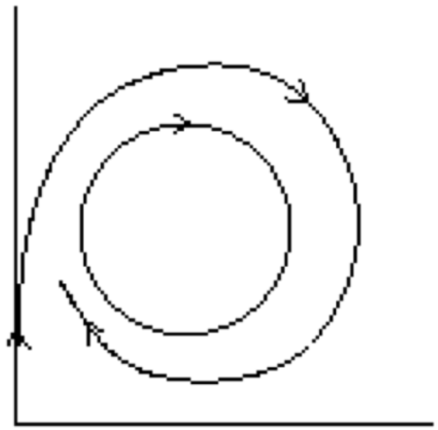
$\beta < 0$ $\beta = 0$ $\beta > 0$

Changing parameter a

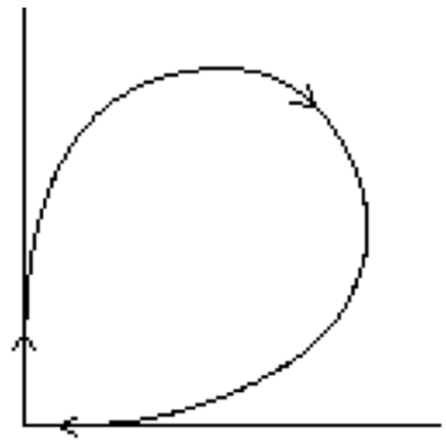


Homoclinic bifurcation

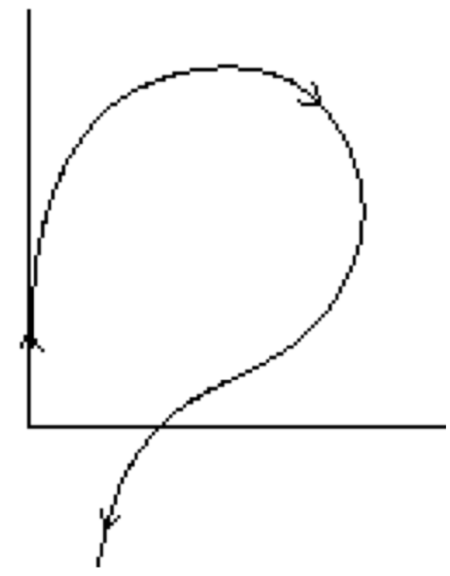
- Collision of a limit cycle with a saddle point



For small parameter values, there is a saddle point at the origin and a limit cycle in the first quadrant.

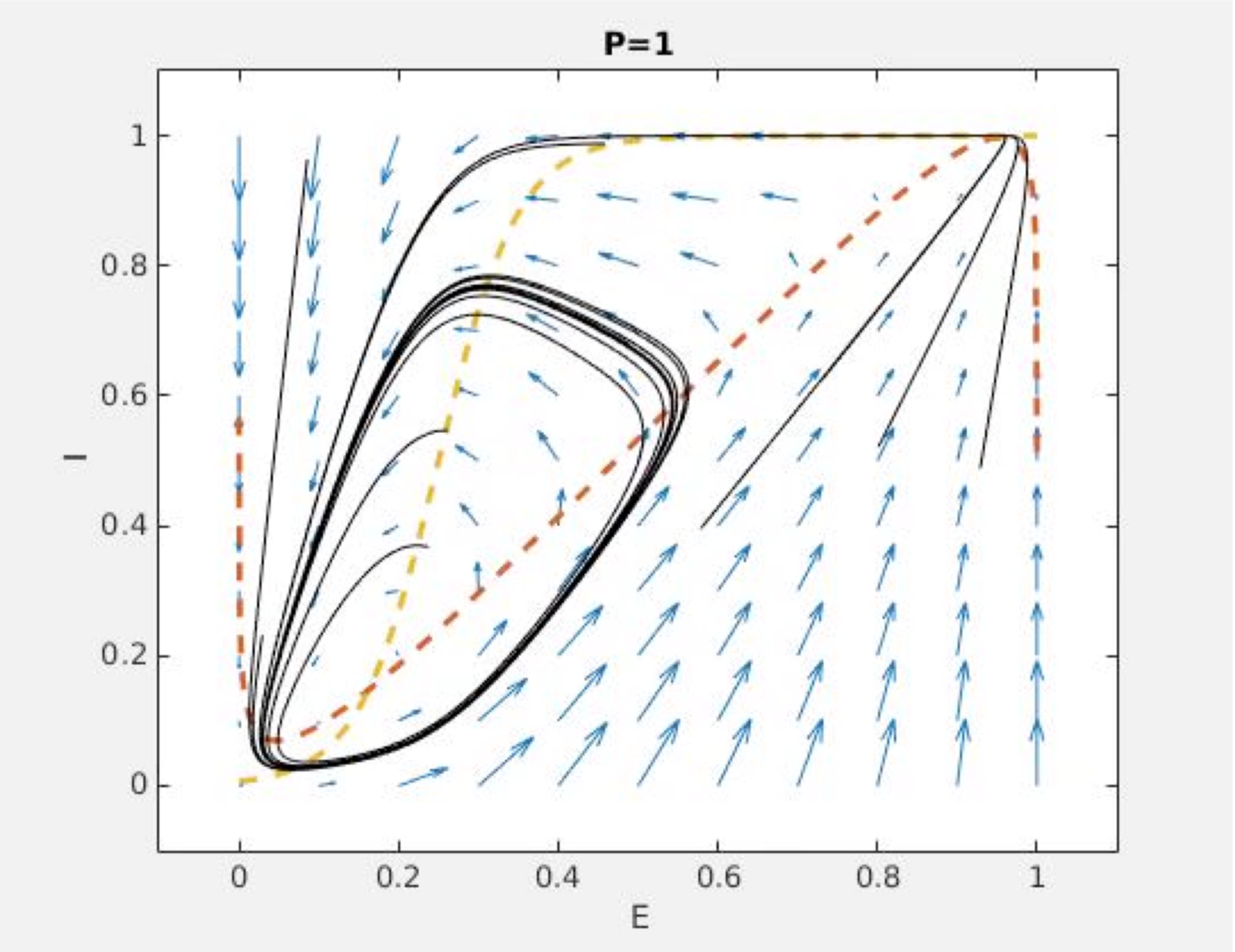


As the bifurcation parameter increases, the limit cycle grows until it exactly intersects the saddle point, yielding an orbit of infinite duration.



When the bifurcation parameter increases further, the limit cycle disappears completely.

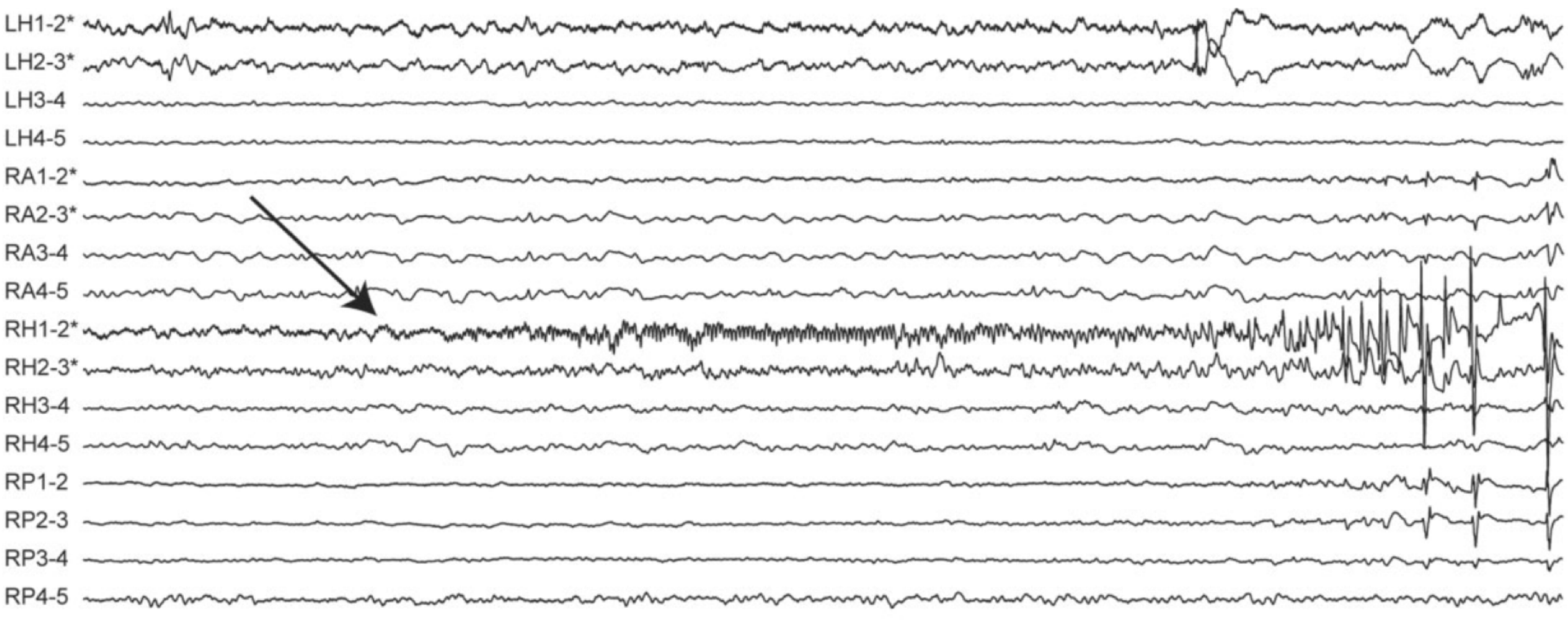
Changing P



Why are bifurcations important?

- Modelling transition:
- Seizure onset
- Epidemics
- Cell cycle transitions

Focal seizure onset



Focal seizure onset

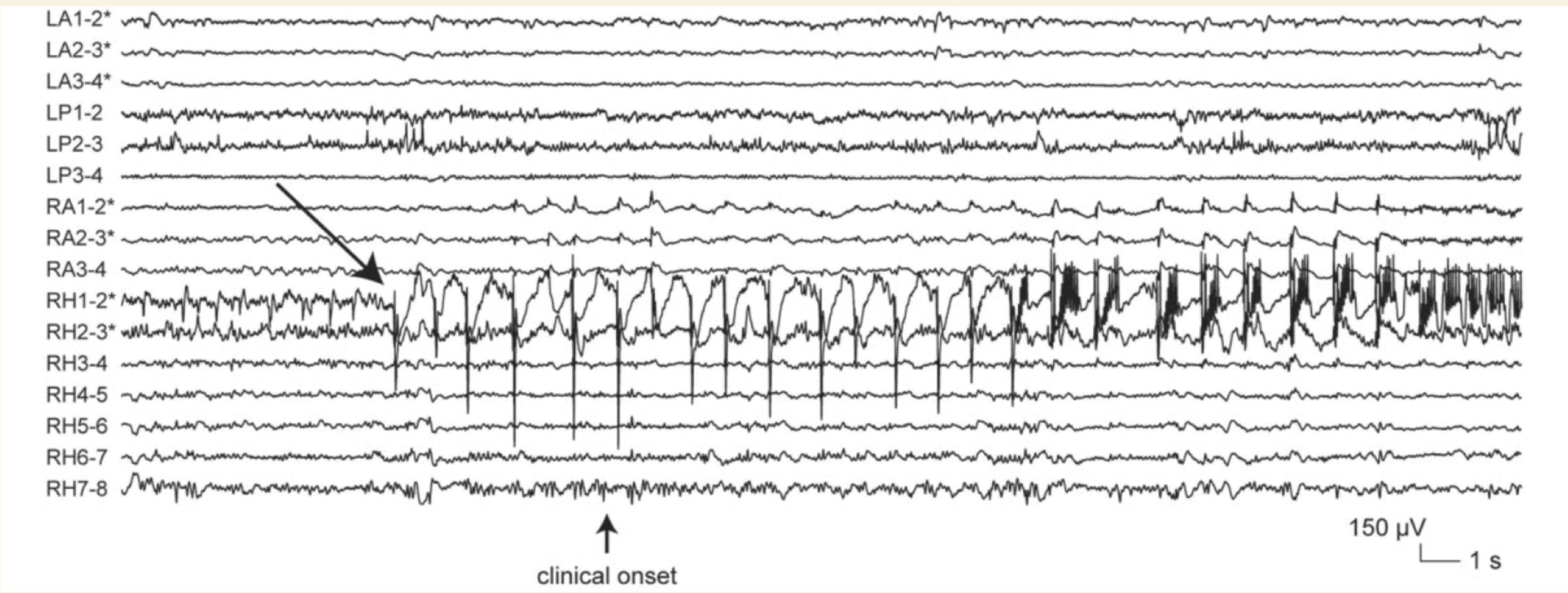
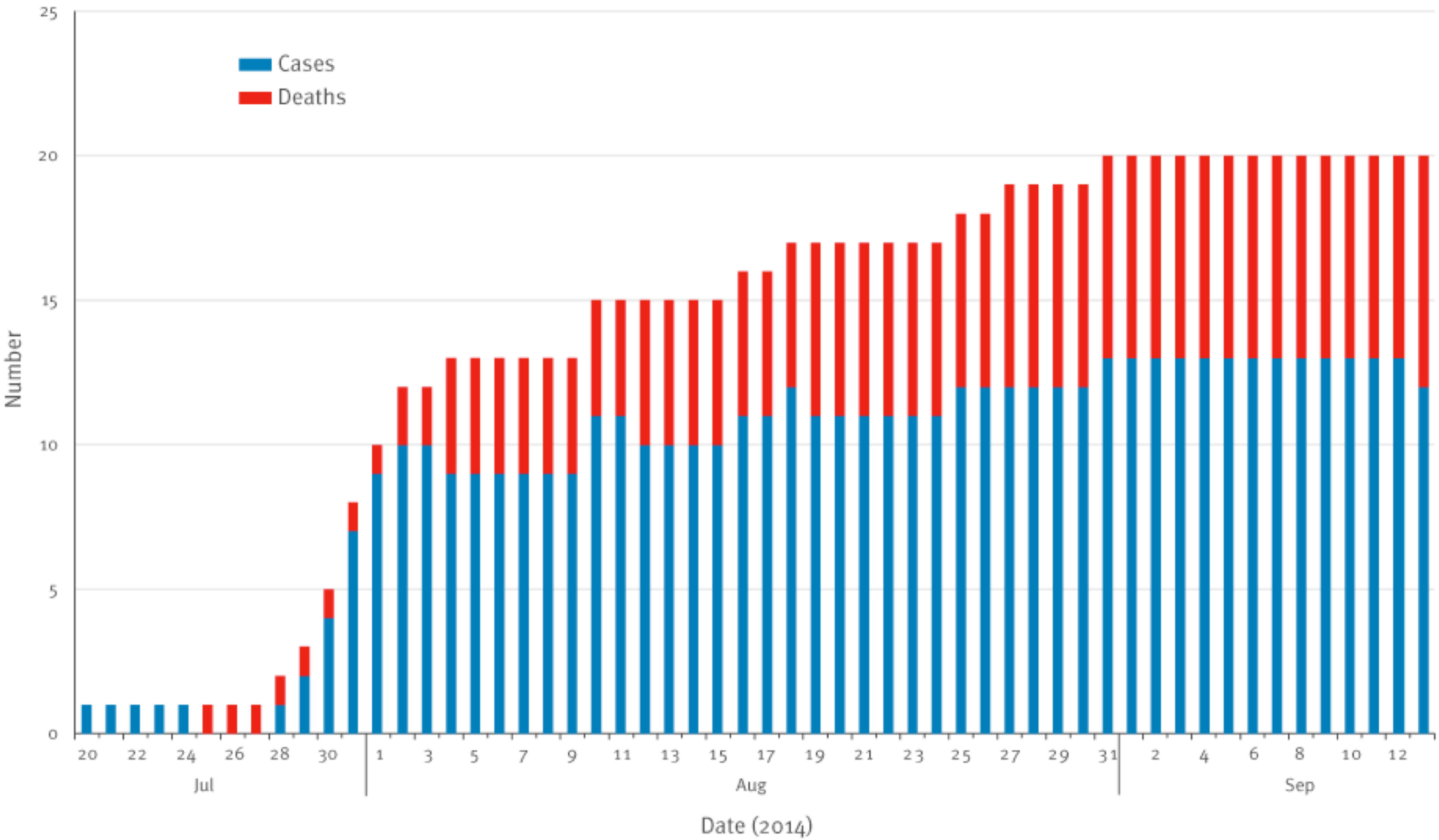


FIGURE 1

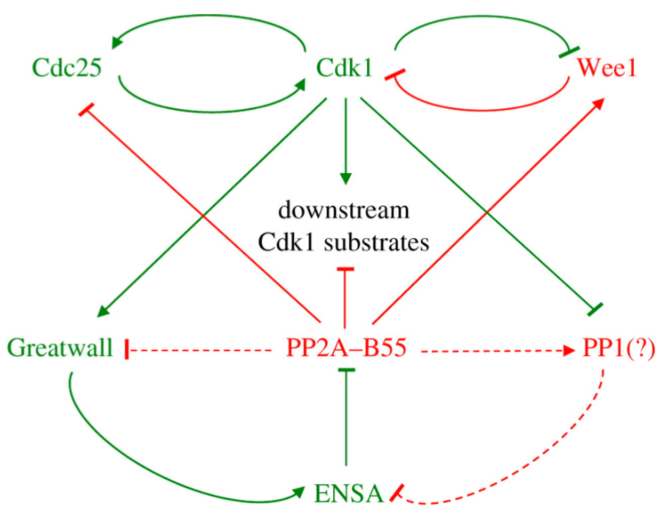
Cumulative reported cases and deaths of Ebola virus disease in Nigeria, July–September 2014

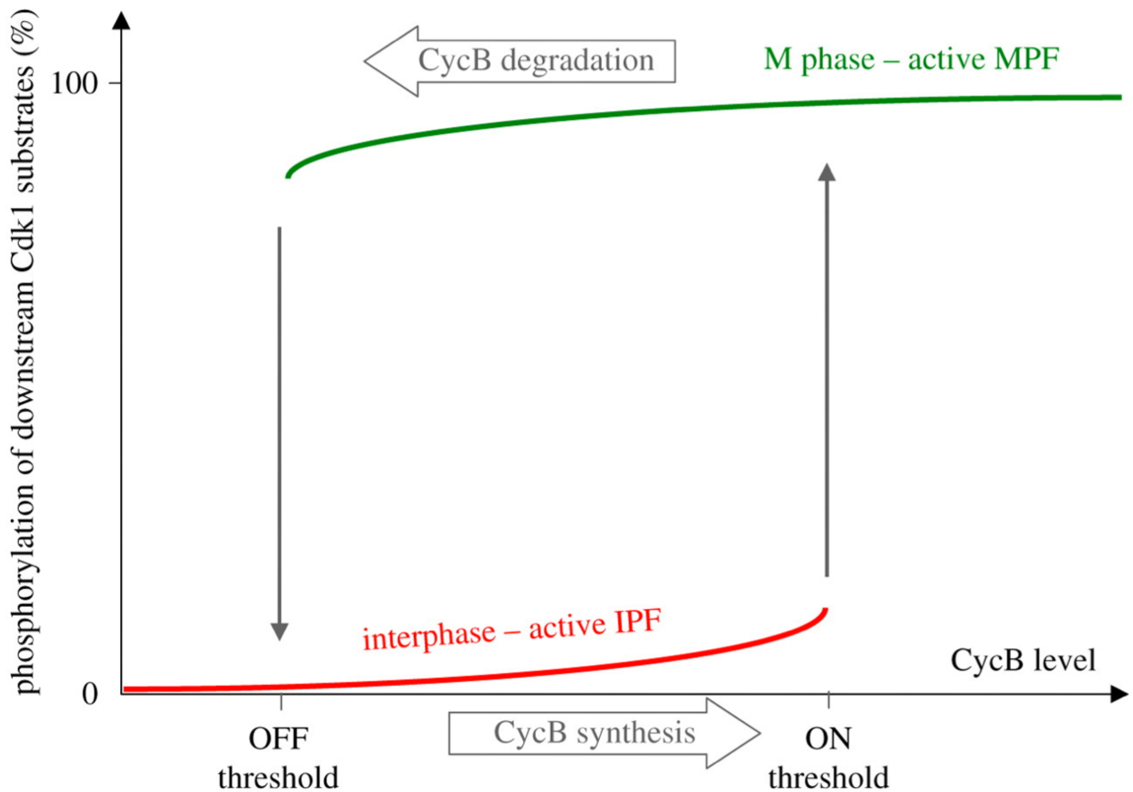


A total of 19 laboratory-confirmed cases, one probable case and eight deaths among the cases have been reported as of 1 October 2014. The index case entered Nigeria on 20 July 2014 and the onset of outbreak is taken from that date.

Source: [1,2,5].

Diagram of Irreversible and Bistable Switch (in mitosis)





Domingo-Sananes et al. Phil Trans R Soc B, 2011. DOI: 10.1098/rstb.2011.0087

How do we analyse bifurcations?

1. By simulation
2. By continuation

By simulation: parameter scan

- Vary a parameter slowly and observe long-term behaviour (in phase space, or as a time series)
- Vary parameter slowly and store information (min/max of an oscillation) about the long term behaviour at each parameter & plot it
- Forward & backward scan to detect bistabilities
- Advantage: Simple and intuitive to understand. Quick way to check systems.
- Drawback: Only stable dynamics can be shown. (Saddles are invisible.)

By numerical continuation

- Uses the mathematical conditions (e.g. eigenvalues) to find the bifurcation points in parameter space
- Available software packages: XPPAuto, MATCONT (matlab package), …
- Advantage: also shows unstable structures
- Drawback: software sometimes fickle and difficult to use. Can be computationally expensive, especially for large systems.

- http://wwwf.imperial.ac.uk/~jswlamb/LDSG/grad0506/files/intro.pdf

Terminology recap

- Limit cycle
- Hopf bifurcation
- Saddle-node bifurcation
- Homoclinic bifurcation
- Continuation

- Comment: Scholarpedia is your best friend in bifurcation theory