# 1 The Basics

This chapter gives a gentle yet concise introduction to most of the terminology used later in the book. Fortunately, much of standard graph theoretic terminology is so intuitive that it is easy to remember; the few terms better understood in their proper setting will be introduced later, when their time has come.

Section 1.1 offers a brief but self-contained summary of the most basic definitions in graph theory, those centred round the notion of a graph. Most readers will have met these definitions before, or will have them explained to them as they begin to read this book. For this reason, Section 1.1 does not dwell on these definitions more than clarity requires: its main purpose is to collect the most basic terms in one place, for easy reference later.

From Section 1.2 onwards, all new definitions will be brought to life almost immediately by a number of simple yet fundamental propositions. Often, these will relate the newly defined terms to one another: the question of how the value of one invariant influences that of another underlies much of graph theory, and it will be good to become familiar with this line of thinking early.

By $\mathbb{N}$ we denote the set of natural numbers, including zero. The set $\mathbb{Z}/n\mathbb{Z}$ of integers modulo $n$ is denoted by $\mathbb{Z}_n$; its elements are written as $\bar{i} := i + n\mathbb{Z}$. When we regard $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ as a field, we also denote it as $\mathbb{F}_2 = \{0, 1\}$. For a real number $x$ we denote by $\lfloor x \rfloor$ the greatest integer $\leqslant x$, and by $\lceil x \rceil$ the least integer $\geqslant x$. Logarithms written as 'log' are taken at base 2; the natural logarithm will be denoted by 'ln'. A set $\mathcal{A} = \{A_1, \dots, A_k\}$ of disjoint subsets of a set $A$ is a *partition* of $A$ if the union $\bigcup \mathcal{A}$ of all the sets $A_i \in \mathcal{A}$ is $A$ and $A_i \neq \emptyset$ for every $i$. Another partition $\{A'_1, \dots, A'_\ell\}$ of $A$ *refines* the partition $\mathcal{A}$ if each $A'_i$ is contained in some $A_j$. By $[A]^k$ we denote the set of all $k$-element subsets of $A$. Sets with $k$ elements will be called *$k$-sets*; subsets with $k$ elements are *$k$-subsets*.

$\mathbb{Z}_n$

$\lfloor x \rfloor, \lceil x \rceil$

log, ln

*partition*

$\bigcup \mathcal{A}$

$[A]^k$

*$k$-set*

## 1.1 Graphs

*graph* *vertex* *edge*

A *graph* is a pair $G = (V, E)$ of sets such that $E \subseteq [V]^2$; thus, the elements of $E$ are 2-element subsets of $V$. To avoid notational ambiguities, we shall always assume tacitly that $V \cap E = \emptyset$. The elements of $V$ are the *vertices* (or *nodes*, or *points*) of the graph $G$, the elements of $E$ are its *edges* (or *lines*). The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by a line if the corresponding two vertices form an edge. Just how these dots and lines are drawn is considered irrelevant: all that matters is the information of which pairs of vertices form an edge and which do not.

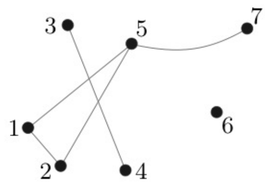


*Fig. 1.1.1.* The graph on $V = \{1, \ldots, 7\}$ with edge set $E = \{\{1,2\},\{1,5\},\{2,5\},\{3,4\},\{5,7\}\}$

*on* $V(G), E(G)$

A graph with vertex set $V$ is said to be a graph *on* $V$. The vertex set of a graph $G$ is referred to as $V(G)$, its edge set as $E(G)$. These conventions are independent of any actual names of these two sets: the vertex set $W$ of a graph $H = (W, F)$ is still referred to as $V(H)$, not as $W(H)$. We shall not always distinguish strictly between a graph and its vertex or edge set. For example, we may speak of a vertex $v \in G$ (rather than $v \in V(G)$), an edge $e \in G$, and so on.

*order* $|G|,\ \|G\|$

The number of vertices of a graph $G$ is its *order*, written as $|G|$; its number of edges is denoted by $\|G\|$. Graphs are *finite*, *infinite*, *countable* and so on according to their order. Except in Chapter 8, our graphs will be finite unless otherwise stated.

$\emptyset$ *trivial graph*

For the *empty graph* $(\emptyset, \emptyset)$ we simply write $\emptyset$. A graph of order 0 or 1 is called *trivial*. Sometimes, e.g. to start an induction, trivial graphs can be useful; at other times they form silly counterexamples and become a nuisance. To avoid cluttering the text with non-triviality conditions, we shall mostly treat the trivial graphs, and particularly the empty graph $\emptyset$, with generous disregard.

*incident* *ends* $E(X,Y)$ $E(v)$

A vertex $v$ is *incident* with an edge $e$ if $v \in e$; then $e$ is an edge *at* $v$. The two vertices incident with an edge are its *endvertices* or *ends*, and an edge *joins* its ends. An edge $\{x, y\}$ is usually written as $xy$ (or $yx$). If $x \in X$ and $y \in Y$, then $xy$ is an $X$–$Y$ *edge*. The set of all $X$–$Y$ edges in a set $E$ is denoted by $E(X, Y)$; instead of $E(\{x\}, Y)$ and $E(X, \{y\})$ we simply write $E(x, Y)$ and $E(X, y)$. The set of all the edges in $E$ at a vertex $v$ is denoted by $E(v)$.

Two vertices $x, y$ of $G$ are *adjacent*, or *neighbours*, if $xy$ is an edge of $G$. Two edges $e \neq f$ are *adjacent* if they have an end in common. If all the vertices of $G$ are pairwise adjacent, then $G$ is *complete*. A complete graph on $n$ vertices is a $K^n$; a $K^3$ is called a *triangle*.

*adjacent* *neighbour* *complete* $K^n$

Pairwise non-adjacent vertices or edges are called *independent*. More formally, a set of vertices or of edges is *independent* if no two of its elements are adjacent. Independent sets of vertices are also called *stable sets*.

*inde-pendent*

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call $G$ and $G'$ *isomorphic*, and write $G \simeq G'$, if there exists a bijection $\varphi: V \to V'$ with $xy \in E \Leftrightarrow \varphi(x)\varphi(y) \in E'$ for all $x, y \in V$. Such a map $\varphi$ is called an *isomorphism*; if $G = G'$, it is called an *automorphism*. We do not normally distinguish between isomorphic graphs. Thus, we usually write $G = G'$ rather than $G \simeq G'$, speak of *the* complete graph on 17 vertices, and so on. If we wish to emphasize that we are only interested in the isomorphism type of a given graph, we informally refer to it as an *abstract graph*.

$\simeq$ *isomor-phism*

A class of graphs that is closed under isomorphism is called a *graph property*. For example, 'containing a triangle' is a graph property: if $G$ contains three pairwise adjacent vertices then so does every graph isomorphic to $G$. A map taking graphs as arguments is called a *graph invariant* if it assigns equal values to isomorphic graphs. The number of vertices and the number of edges of a graph are two simple graph invariants; the greatest number of pairwise adjacent vertices is another.

*property* *invariant*

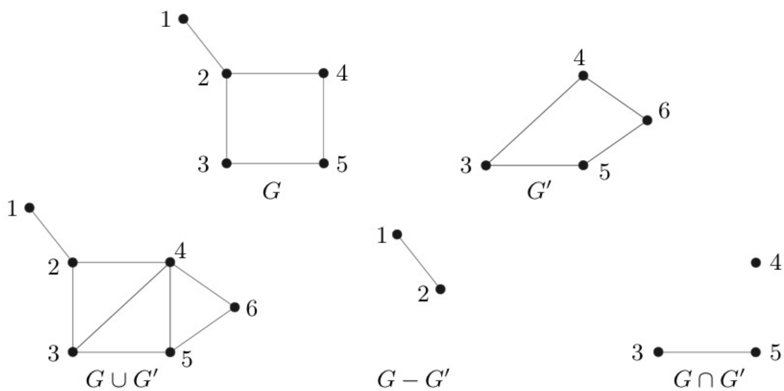


*Fig. 1.1.2.* Union, difference and intersection; the vertices 2,3,4 induce (or span) a triangle in $G \cup G'$ but not in $G$

We set $G \cup G' := (V \cup V', E \cup E')$ and $G \cap G' := (V \cap V', E \cap E')$. If $G \cap G' = \emptyset$, then $G$ and $G'$ are *disjoint*. If $V' \subseteq V$ and $E' \subseteq E$, then $G'$ is a *subgraph* of $G$ (and $G$ a *supergraph* of $G'$), written as $G' \subseteq G$.

$G \cap G'$ *subgraph* $G' \subseteq G$

Less formally, we say that $G$ *contains* $G'$. If $G' \subseteq G$ and $G' \neq G$, then $G'$ is a *proper subgraph* of $G$.

*induced subgraph*

If $G' \subseteq G$ and $G'$ contains *all* the edges $xy \in E$ with $x, y \in V'$, then $G'$ is an *induced subgraph* of $G$; we say that $V'$ *induces* or *spans* $G'$ in $G$,

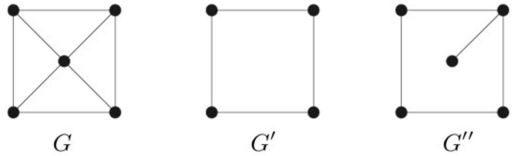


*Fig. 1.1.3.* A graph $G$ with subgraphs $G'$ and $G''$: $G'$ is an induced subgraph of $G$, but $G''$ is not

*G[U]*

and write $G' =: G[V']$. Thus if $U \subseteq V$ is any set of vertices, then $G[U]$ denotes the graph on $U$ whose edges are precisely the edges of $G$ with both ends in $U$. If $H$ is a subgraph of $G$, not necessarily induced, we abbreviate $G[V(H)]$ to $G[H]$. Finally, $G' \subseteq G$ is a *spanning* subgraph of $G$ if $V'$ spans all of $G$, i.e. if $V' = V$.

*spanning*

*−*

If $U$ is any set of vertices (usually of $G$), we write $G - U$ for $G[V \setminus U]$. In other words, $G - U$ is obtained from $G$ by *deleting* all the vertices in $U \cap V$ and their incident edges. If $U = \{v\}$ is a singleton, we write $G - v$ rather than $G - \{v\}$. Instead of $G - V(G')$ we simply write $G - G'$. For a subset $F$ of $[V]^2$ we write $G - F := (V, E \setminus F)$ and $G + F := (V, E \cup F)$; as above, $G - \{e\}$ and $G + \{e\}$ are abbreviated to $G - e$ and $G + e$. We call $G$ *edge-maximal* with a given graph property if $G$ itself has the property but no graph $G + xy$ does, for non-adjacent vertices $x, y \in G$.

*+*

*edge-maximal*

*minimal*

*maximal*

More generally, when we call a graph *minimal* or *maximal* with some property but have not specified any particular ordering, we are referring to the subgraph relation. When we speak of minimal or maximal sets of vertices or edges, the reference is simply to set inclusion.

*G ∗ G′*

*complement $\overline{G}$*

*line graph L(G)*

If $G$ and $G'$ are disjoint, we denote by $G * G'$ the graph obtained from $G \cup G'$ by joining all the vertices of $G$ to all the vertices of $G'$. For example, $K^2 * K^3 = K^5$. The *complement* $\overline{G}$ of $G$ is the graph on $V$ with edge set $[V]^2 \setminus E$. The *line graph* $L(G)$ of $G$ is the graph on $E$ in which $x, y \in E$ are adjacent as vertices if and only if they are adjacent as edges in $G$.

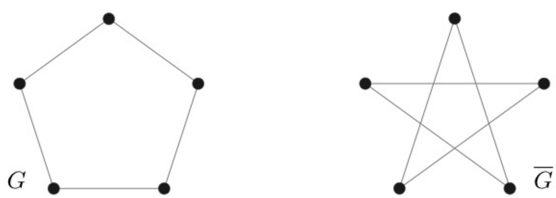


*Fig. 1.1.4.* A graph isomorphic to its complement

## 1.2 The degree of a vertex

Let $G = (V, E)$ be a (non-empty) graph. The set of neighbours of a vertex $v$ in $G$ is denoted by $N_G(v)$, or briefly by $N(v)$.[1] More generally for $U \subseteq V$, the neighbours in $V \setminus U$ of vertices in $U$ are called *neighbours of* $U$; their set is denoted by $N(U)$. $N(v)$

The *degree* (or *valency*) $d_G(v) = d(v)$ of a vertex $v$ is the number $|E(v)|$ of edges at $v$; by our definition of a graph,[2] this is equal to the number of neighbours of $v$. A vertex of degree 0 is *isolated*. The number $\delta(G) := \min \{ d(v) \mid v \in V \}$ is the *minimum degree* of $G$, the number $\Delta(G) := \max \{ d(v) \mid v \in V \}$ its *maximum degree*. If all the vertices of $G$ have the same degree $k$, then $G$ is *$k$-regular*, or simply *regular*. A 3-regular graph is called *cubic*. *degree* $d(v)$ *isolated* $\delta(G)$ $\Delta(G)$ *regular* *cubic*

The number

$$d(G) := \frac{1}{|V|} \sum_{v \in V} d(v)$$

$d(G)$

is the *average degree* of $G$. Clearly, *average degree*

$$\delta(G) \leqslant d(G) \leqslant \Delta(G)\,.$$

The average degree quantifies globally what is measured locally by the vertex degrees: the number of edges of $G$ per vertex. Sometimes it will be convenient to express this ratio directly, as $\varepsilon(G) := |E|/|V|$. $\varepsilon(G)$

The quantities $d$ and $\varepsilon$ are, of course, intimately related. Indeed, if we sum up all the vertex degrees in $G$, we count every edge exactly twice: once from each of its ends. Thus

$$|E| = \tfrac{1}{2} \sum_{v \in V} d(v) = \tfrac{1}{2} d(G) \cdot |V|\,,$$

and therefore

$$\varepsilon(G) = \tfrac{1}{2} d(G)\,.$$

**Proposition 1.2.1.** *The number of vertices of odd degree in a graph is always even.* [10.3.1]

*Proof.* A graph on $V$ has $\frac{1}{2} \sum_{v \in V} d(v)$ edges, so $\sum d(v)$ is an even number. □

[1] Here, as elsewhere, we drop the index referring to the underlying graph if the reference is clear.

[2] but not for multigraphs; see Section 1.10

If a graph has large minimum degree, i.e. everywhere, locally, many edges per vertex, it also has many edges per vertex globally: $\varepsilon(G) = \frac{1}{2}d(G) \geqslant \frac{1}{2}\delta(G)$. Conversely, of course, its average degree may be large even when its minimum degree is small. However, the vertices of large degree cannot be scattered completely among vertices of small degree: as the next proposition shows, every graph $G$ has a subgraph whose average degree is no less than the average degree of $G$, and whose minimum degree is more than half its average degree:

[1.4.3] [3.5.1] **Proposition 1.2.2.** *Every graph $G$ with at least one edge has a subgraph $H$ with $\delta(H) > \varepsilon(H) \geqslant \varepsilon(G)$.*

*Proof.* To construct $H$ from $G$, let us try to delete vertices of small degree one by one, until only vertices of large degree remain. Up to which degree $d(v)$ can we afford to delete a vertex $v$, without lowering $\varepsilon$? Clearly, up to $d(v) = \varepsilon$: then the number of vertices decreases by 1 and the number of edges by at most $\varepsilon$, so the overall ratio $\varepsilon$ of edges to vertices will not decrease.

Formally, we construct a sequence $G = G_0 \supseteq G_1 \supseteq \ldots$ of induced subgraphs of $G$ as follows. If $G_i$ has a vertex $v_i$ of degree $d(v_i) \leqslant \varepsilon(G_i)$, we let $G_{i+1} := G_i - v_i$; if not, we terminate our sequence and set $H := G_i$. By the choices of $v_i$ we have $\varepsilon(G_{i+1}) \geqslant \varepsilon(G_i)$ for all $i$, and hence $\varepsilon(H) \geqslant \varepsilon(G)$.

What else can we say about the graph $H$? Since $\varepsilon(K^1) = 0 < \varepsilon(G)$, none of the graphs in our sequence is trivial, so in particular $H \neq \emptyset$. The fact that $H$ has no vertex suitable for deletion thus implies $\delta(H) > \varepsilon(H)$, as claimed. □

## 1.3 Paths and cycles

*path* A *path* is a non-empty graph $P = (V, E)$ of the form

$$V = \{x_0, x_1, \ldots, x_k\} \qquad E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\},$$

where the $x_i$ are all distinct. The vertices $x_0$ and $x_k$ are *linked* by $P$ and are called its *endvertices* or *ends*; the vertices $x_1, \ldots, x_{k-1}$ are the *inner*
*length* vertices of $P$. The number of edges of a path is its *length*, and the path
$P^k$ of length $k$ is denoted by $P^k$. Note that $k$ is allowed to be zero; thus, $P^0 = K^1$.

We often refer to a path by the natural sequence of its vertices,[3]

[3] More precisely, by one of the two natural sequences: $x_0 \ldots x_k$ and $x_k \ldots x_0$ denote the same path. Still, it often helps to fix one of these two orderings of $V(P)$ notationally: we may then speak of things like the 'first' vertex on $P$ with a certain property, etc.

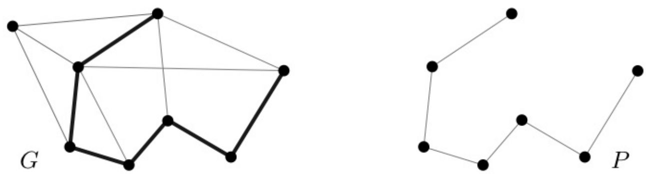


*Fig. 1.3.1.* A path $P = P^6$ in $G$

writing, say, $P = x_0x_1 \dots x_k$ and calling $P$ a path *from* $x_0$ *to* $x_k$ (as well as *between* $x_0$ and $x_k$).

*$xPy$, $\mathring{P}$*

For $0 \leqslant i \leqslant j \leqslant k$ we write

$$\begin{aligned} Px_i &:= x_0 \dots x_i \\ x_iP &:= x_i \dots x_k \\ x_iPx_j &:= x_i \dots x_j \end{aligned}$$

and

$$\begin{aligned} \mathring{P} &:= x_1 \dots x_{k-1} \\ P\mathring{x}_i &:= x_0 \dots x_{i-1} \\ \mathring{x}_iP &:= x_{i+1} \dots x_k \\ \mathring{x}_iP\mathring{x}_j &:= x_{i+1} \dots x_{j-1} \end{aligned}$$

for the appropriate subpaths of $P$. We use similar intuitive notation for the concatenation of paths; for example, if the union $Px \cup xQy \cup yR$ of three paths is again a path, we may simply denote it by $PxQyR$.

*$PxQyR$*

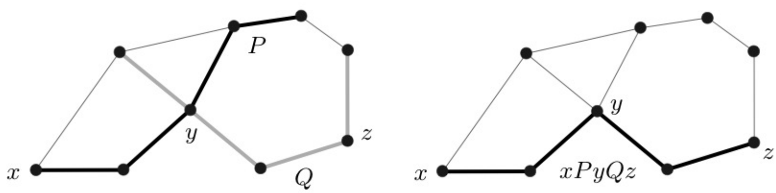


*Fig. 1.3.2.* Paths $P$, $Q$ and $xPyQz$

*$A$–$B$ path*

*independent*

Given sets $A, B$ of vertices, we call $P = x_0 \dots x_k$ an *$A$–$B$ path* if $V(P) \cap A = \{x_0\}$ and $V(P) \cap B = \{x_k\}$. As before, we write $a$–$B$ path rather than $\{a\}$–$B$ path, etc. Two or more paths are *independent* if none of them contains an inner vertex of another. Two $a$–$b$ paths, for instance, are independent if and only if $a$ and $b$ are their only common vertices.

*$H$-path*

Given a graph $H$, we call $P$ an *$H$-path* if $P$ is non-trivial and meets $H$ exactly in its ends. In particular, the edge of any $H$-path of length 1 is never an edge of $H$.

*cycle* · *length* · $C^k$

If $P = x_0 \dots x_{k-1}$ is a path and $k \geqslant 3$, then the graph $C := P + x_{k-1}x_0$ is called a *cycle*. As with paths, we often denote a cycle by its (cyclic) sequence of vertices; the above cycle $C$ might be written as $x_0 \dots x_{k-1}x_0$. The *length* of a cycle is its number of edges (or vertices); the cycle of length $k$ is called a *k-cycle* and denoted by $C^k$.

*girth* $g(G)$ · *circumference* · *chord* · *induced cycle*

The minimum length of a cycle (contained) in a graph $G$ is the *girth* $g(G)$ of $G$; the maximum length of a cycle in $G$ is its *circumference*. (If $G$ does not contain a cycle, we set the former to $\infty$, the latter to zero.) An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a *chord* of that cycle. Thus, an *induced cycle* in $G$, a cycle in $G$ forming an induced subgraph, is one that has no chords (Fig. 1.3.3).

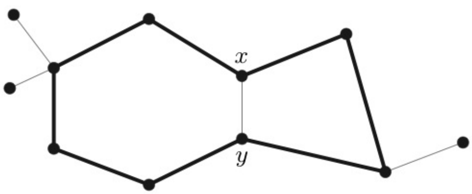


*Fig. 1.3.3.* A cycle $C^8$ with chord $xy$, and induced cycles $C^6, C^4$

If a graph has large minimum degree, it contains long paths and cycles (see also Exercise 8):

[1.4.3]
[3.5.1]

**Proposition 1.3.1.** *Every graph $G$ contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$ (provided that $\delta(G) \geqslant 2$).*

*Proof.* Let $x_0 \dots x_k$ be a longest path in $G$. Then all the neighbours of $x_k$ lie on this path (Fig. 1.3.4). Hence $k \geqslant d(x_k) \geqslant \delta(G)$. If $i < k$ is minimal with $x_i x_k \in E(G)$, then $x_i \dots x_k x_i$ is a cycle of length at least $\delta(G)+1$. $\square$

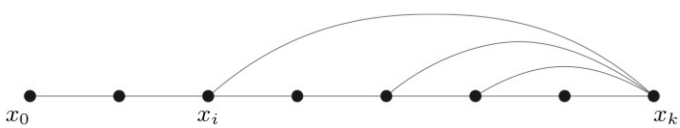


*Fig. 1.3.4.* A longest path $x_0 \dots x_k$, and the neighbours of $x_k$

Minimum degree and girth, on the other hand, are not related (unless we fix the number of vertices): as we shall see in Chapter 11, there are graphs combining arbitrarily large minimum degree with arbitrarily large girth.

*distance* $d(x,y)$ · *diameter* $\operatorname{diam} G$

The *distance* $d_G(x,y)$ in $G$ of two vertices $x, y$ is the length of a shortest $x$–$y$ path in $G$; if no such path exists, we set $d(x,y) := \infty$. The greatest distance between any two vertices in $G$ is the *diameter* of $G$, denoted by $\operatorname{diam} G$. Diameter and girth are, of course, related:

**Proposition 1.3.2.** *Every graph $G$ containing a cycle satisfies $g(G) \leqslant 2 \operatorname{diam} G + 1$.*

*Proof.* Let $C$ be a shortest cycle in $G$. If $g(G) \geqslant 2 \operatorname{diam} G + 2$, then $C$ has two vertices whose distance in $C$ is at least $\operatorname{diam} G + 1$. In $G$, these vertices have a lesser distance; any shortest path $P$ between them is therefore not a subgraph of $C$. Thus, $P$ contains a $C$-path $xPy$. Together with the shorter of the two $x$–$y$ paths in $C$, this path $xPy$ forms a shorter cycle than $C$, a contradiction. □

A vertex is *central* in $G$ if its greatest distance from any other vertex is as small as possible. This distance is the *radius* of $G$, denoted by $\operatorname{rad} G$. Thus, formally, $\operatorname{rad} G = \min_{x \in V(G)} \max_{y \in V(G)} d_G(x, y)$. As one easily checks (exercise), we have central radius $\operatorname{rad} G$

$$\operatorname{rad} G \leqslant \operatorname{diam} G \leqslant 2 \operatorname{rad} G\,.$$

Diameter and radius are not related to minimum, average or maximum degree if we say nothing about the order of the graph. However, graphs of large diameter and minimum degree must be large (larger than forced by each of the two parameters alone; see Exercise 9), and graphs of small diameter and maximum degree must be small:

**Proposition 1.3.3.** *A graph $G$ of radius at most $k$ and maximum degree at most $d \geqslant 3$ has fewer than $\frac{d}{d-2}(d-1)^k$ vertices.* [9.4.1] [9.4.2]

*Proof.* Let $z$ be a central vertex in $G$, and let $D_i$ denote the set of vertices of $G$ at distance $i$ from $z$. Then $V(G) = \bigcup_{i=0}^{k} D_i$. Clearly $|D_0| = 1$ and $|D_1| \leqslant d$. For $i \geqslant 1$ we have $|D_{i+1}| \leqslant (d-1)|D_i|$, because every vertex in $D_{i+1}$ is a neighbour of a vertex in $D_i$ (why?), and each vertex in $D_i$ has at most $d-1$ neighbours in $D_{i+1}$ (since it has another neighbour in $D_{i-1}$). Thus $|D_{i+1}| \leqslant d(d-1)^i$ for all $i < k$ by induction, giving

$$|G| \leqslant 1 + d \sum_{i=0}^{k-1} (d-1)^i = 1 + \frac{d}{d-2}\big((d-1)^k - 1\big) < \frac{d}{d-2}(d-1)^k.$$

□

Similarly, we can bound the order of $G$ from below by assuming that both its minimum degree and girth are large. For $d \in \mathbb{R}$ and $g \in \mathbb{N}$ let

$$n_0(d, g) := \begin{cases} 1 + d \displaystyle\sum_{i=0}^{r-1} (d-1)^i & \text{if } g =: 2r+1 \text{ is odd;} \\ 2 \displaystyle\sum_{i=0}^{r-1} (d-1)^i & \text{if } g =: 2r \text{ is even.} \end{cases}$$

It is not difficult to prove that a graph of minimum degree $\delta$ and girth $g$ has at least $n_0(\delta, g)$ vertices (Exercise 7). Interestingly, one can obtain the same bound for its average degree:

**Theorem 1.3.4.** (Alon, Hoory & Linial 2002)
*Let $G$ be a graph. If $d(G) \geqslant d \geqslant 2$ and $g(G) \geqslant g \in \mathbb{N}$ then $|G| \geqslant n_0(d, g)$.*

One aspect of Theorem 1.3.4 is that it guarantees the existence of a short cycle compared with $|G|$. Using just the easy minimum degree version of Exercise 7, we get the following rather general bound:

[2.3.1] **Corollary 1.3.5.** *If $\delta(G) \geqslant 3$ then $g(G) < 2 \log |G|$.*

*Proof.* If $g := g(G)$ is even then

$$n_0(3, g) = 2\,\frac{2^{g/2} - 1}{2 - 1} = 2^{g/2} + (2^{g/2} - 2) > 2^{g/2},$$

while if $g$ is odd then

$$n_0(3, g) = 1 + 3\,\frac{2^{(g-1)/2} - 1}{2 - 1} = \frac{3}{\sqrt{2}}\,2^{g/2} - 2 > 2^{g/2}.$$

As $|G| \geqslant n_0(3, g)$, the result follows. □

*walk* A *walk* (of *length* $k$) in a graph $G$ is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \ldots e_{k-1} v_k$ of vertices and edges in $G$ such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If $v_0 = v_k$, the walk is *closed*. If the vertices in a walk are all distinct, it defines an obvious path in $G$. In general, every walk between two vertices contains[4] a path between these vertices (proof?).

## 1.4 Connectivity

*connected* A non-empty graph $G$ is called *connected* if any two of its vertices are linked by a path in $G$. If $U \subseteq V(G)$ and $G[U]$ is connected, we also call $U$ itself connected (in $G$). Instead of 'not connected' we usually say 'disconnected'.

[1.5.2] **Proposition 1.4.1.** *The vertices of a connected graph $G$ can always be enumerated, say as $v_1, \ldots, v_n$, so that $G_i := G[v_1, \ldots, v_i]$ is connected for every $i$.*

[4] We shall often use terms defined for graphs also for walks, as long as their meaning is obvious.

*Proof.* Pick any vertex as $v_1$, and assume inductively that $v_1, \ldots, v_i$ have been chosen for some $i < |G|$. Now pick a vertex $v \in G - G_i$. As $G$ is connected, it contains a $v$–$v_1$ path $P$. Choose as $v_{i+1}$ the last vertex of $P$ in $G - G_i$; then $v_{i+1}$ has a neighbour in $G_i$. The connectedness of every $G_i$ follows by induction on $i$. □

*component*

Let $G = (V, E)$ be a graph. A maximal connected subgraph of $G$ is a *component* of $G$. Clearly, the components are induced subgraphs, and their vertex sets partition $V$. Since connected graphs are non-empty, the empty graph has no components.

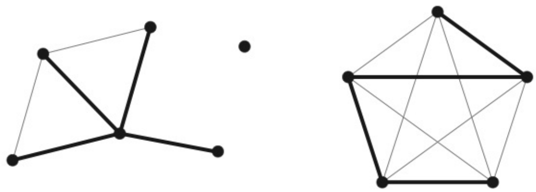

*Fig. 1.4.1.* A graph with three components, and a minimal spanning connected subgraph in each component

*separate*

*separator*

*cutvertex*

*bridge*

If $A, B \subseteq V$ and $X \subseteq V \cup E$ are such that every $A$–$B$ path in $G$ contains a vertex or an edge from $X$, we say that $X$ *separates* the sets $A$ and $B$ in $G$. Note that this implies $A \cap B \subseteq X$. More generally we say that $X$ *separates* $G$ if $G - X$ is disconnected, that is, if $X$ separates in $G$ some two vertices that are not in $X$. A separating set of vertices is a *separator*. Separating sets of edges have no generic name, but some such sets do; see Section 1.9 for the definition of *cuts* and *bonds*. A vertex which separates two other vertices of the same component is a *cutvertex*, and an edge separating its ends is a *bridge*. Thus, the bridges in a graph are precisely those edges that do not lie on any cycle.

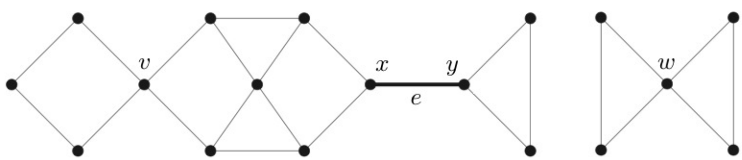


*Fig. 1.4.2.* A graph with cutvertices $v, x, y, w$ and bridge $e = xy$

*separation*

The unordered pair $\{A, B\}$ is a *separation* of $G$ if $A \cup B = V$ and $G$ has no edge between $A \setminus B$ and $B \setminus A$. Clearly, the latter is equivalent to saying that $A \cap B$ separates $A$ from $B$. If both $A \setminus B$ and $B \setminus A$ are non-empty, the separation is *proper*. The number $|A \cap B|$ is the *order* of the separation $\{A, B\}$.

*k-connected*

$G$ is called *$k$-connected* (for $k \in \mathbb{N}$) if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. In other words, no two vertices of $G$

are separated by fewer than $k$ other vertices. Every (non-empty) graph is 0-connected, and the 1-connected graphs are precisely the non-trivial connected graphs. The greatest integer $k$ such that $G$ is $k$-connected is the *connectivity* $\kappa(G)$ of $G$. Thus, $\kappa(G) = 0$ if and only if $G$ is disconnected or a $K^1$, and $\kappa(K^n) = n-1$ for all $n \geqslant 1$.

*connectivity* $\kappa(G)$

If $|G| > 1$ and $G - F$ is connected for every set $F \subseteq E$ of fewer than $\ell$ edges, then $G$ is called *$\ell$-edge-connected*. The greatest integer $\ell$ such that $G$ is $\ell$-edge-connected is the *edge-connectivity* $\lambda(G)$ of $G$. In particular, we have $\lambda(G) = 0$ if $G$ is disconnected.

*$\ell$-edge-connected*
*edge-connectivity* $\lambda(G)$

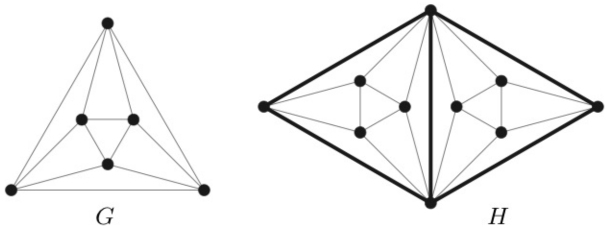


*Fig. 1.4.3.* The *octahedron* $G$ (left) with $\kappa(G) = \lambda(G) = 4$, and a graph $H$ with $\kappa(H) = 2$ but $\lambda(H) = 4$

**Proposition 1.4.2.** *If $G$ is non-trivial then $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$.*

*Proof.* The second inequality follows from the fact that all the edges incident with a fixed vertex separate $G$. To prove the first, let $F$ be a set of $\lambda(G)$ edges such that $G - F$ is disconnected. Such a set exists by definition of $\lambda$; note that $F$ is a minimal separating set of edges in $G$. We show that $\kappa(G) \leqslant |F|$.

Suppose first that $G$ has a vertex $v$ that is not incident with an edge in $F$. Let $C$ be the component of $G - F$ containing $v$. Then the vertices of $C$ that are incident with an edge in $F$ separate $v$ from $G - C$. Since no edge in $F$ has both ends in $C$ (by the minimality of $F$), there are at most $|F|$ such vertices, giving $\kappa(G) \leqslant |F|$ as desired.

Suppose now that every vertex is incident with an edge in $F$. Let $v$ be any vertex, and let $C$ be the component of $G - F$ containing $v$. Then the neighbours $w$ of $v$ with $vw \notin F$ lie in $C$ and are incident with distinct edges in $F$ (again by the minimality of $F$), giving $d_G(v) \leqslant |F|$. As $N_G(v)$ separates $v$ from any other vertices in $G$, this yields $\kappa(G) \leqslant |F|$—unless there are no other vertices, i.e. unless $\{v\} \cup N(v) = V$. But $v$ was an arbitrary vertex. So we may assume that $G$ is complete, giving $\kappa(G) = \lambda(G) = |G| - 1$. □

By Proposition 1.4.2, high connectivity requires a large minimum degree. Conversely, large minimum degree does not ensure high connectivity, not even high edge-connectivity (examples?). It does, however, imply the existence of a highly connected subgraph:

**Theorem 1.4.3.** (Mader 1972) [7.2.1] [11.2.3]
*Let $0 \neq k \in \mathbb{N}$. Every graph $G$ with $d(G) \geqslant 4k$ has a $(k+1)$-connected subgraph $H$ such that $\varepsilon(H) > \varepsilon(G) - k$.*

*Proof.* Put $\gamma := \varepsilon(G)$ $(\geqslant 2k)$, and consider the subgraphs $G' \subseteq G$ such that (1.2.2) (1.3.1) $\gamma$

$$|G'| \geqslant 2k \quad \text{and} \quad \|G'\| > \gamma \left(|G'| - k\right). \qquad (*)$$

Such graphs $G'$ exist since $G$ is one; let $H$ be one of smallest order. $H$

No graph $G'$ as in $(*)$ can have order exactly $2k$, since this would imply that $\|G'\| > \gamma k \geqslant 2k^2 > \binom{|G'|}{2}$. The minimality of $H$ therefore implies that $\delta(H) > \gamma$: otherwise we could delete a vertex of degree at most $\gamma$ and obtain a graph $G' \subsetneq H$ still satisfying $(*)$. In particular, we have $|H| \geqslant \gamma$. Dividing the inequality of $\|H\| > \gamma\,|H| - \gamma k$ from $(*)$ by $|H|$ therefore yields $\varepsilon(H) > \gamma - k$, as desired.

It remains to show that $H$ is $(k+1)$-connected. If not, then $H$ has a proper separation $\{U_1, U_2\}$ of order at most $k$; put $H[U_i] =: H_i$. $H_1, H_2$ Since any vertex $v \in U_1 \smallsetminus U_2$ has all its $d(v) \geqslant \delta(H) > \gamma$ neighbours from $H$ in $H_1$, we have $|H_1| \geqslant \gamma \geqslant 2k$. Similarly, $|H_2| \geqslant 2k$. As by the minimality of $H$ neither $H_1$ nor $H_2$ satisfies $(*)$, we further have

$$\|H_i\| \leqslant \gamma \left(|H_i| - k\right)$$

for $i = 1, 2$. But then

$$\begin{aligned} \|H\| &\leqslant \|H_1\| + \|H_2\| \\ &\leqslant \gamma \left(|H_1| + |H_2| - 2k\right) \\ &\leqslant \gamma \left(|H| - k\right) \qquad (\text{as } |H_1 \cap H_2| \leqslant k), \end{aligned}$$

which contradicts $(*)$ for $H$. □

## 1.5 Trees and forests

An *acyclic* graph, one not containing any cycles, is called a *forest*. A connected forest is called a *tree*. (Thus, a forest is a graph whose components are trees.) The vertices of degree 1 in a tree are its *leaves*.[5] Every nontrivial tree has a leaf—consider, for example, the ends of a longest path. This little fact often comes in handy, especially in induction proofs about trees: if we remove a leaf from a tree, what remains is still a tree. *forest* *tree* *leaf*

[5] . . . except that the *root* of a tree (see below) is never called a leaf, even if it has degree 1.

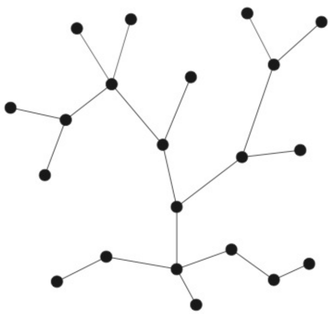

*Fig. 1.5.1.* A tree

[1.6.1] [1.9.5] [4.2.9] **Theorem 1.5.1.** *The following assertions are equivalent for a graph $T$:*

(i) *$T$ is a tree;*

(ii) *Any two vertices of $T$ are linked by a unique path in $T$;*

(iii) *$T$ is minimally connected, i.e. $T$ is connected but $T - e$ is disconnected for every edge $e \in T$;*

(iv) *$T$ is maximally acyclic, i.e. $T$ contains no cycle but $T + xy$ does, for any two non-adjacent vertices $x, y \in T$.* □

The proof of Theorem 1.5.1 is straightforward, and a good exercise for anyone not yet familiar with all the notions it relates. Extending our notation for paths from Section 1.3, we write $xTy$ for the unique path in a tree $T$ between two vertices $x, y$ (see (ii) above).

*$xTy$*

A common application of Theorem 1.5.1 is that every connected graph contains a spanning tree: take a minimal connected spanning subgraph and use (iii), or take a maximal acyclic subgraph and apply (iv). Figure 1.4.1 shows a spanning tree in each of the three components of the graph depicted. When $T$ is a spanning tree of $G$, the edges in $E(G) \setminus E(T)$ are the *chords* of $T$ in $G$.

*chord*

**Corollary 1.5.2.** *The vertices of a tree can always be enumerated, say as $v_1, \ldots, v_n$, so that every $v_i$ with $i \geqslant 2$ has a unique neighbour in $\{v_1, \ldots, v_{i-1}\}$.*

(1.4.1) *Proof.* Use the enumeration from Proposition 1.4.1. □

[1.9.5] [2.4.1] [2.4.4] [4.2.9] **Corollary 1.5.3.** *A connected graph with $n$ vertices is a tree if and only if it has $n - 1$ edges.*

*Proof.* Induction on $i$ shows that the subgraph spanned by the first $i$ vertices in Corollary 1.5.2 has $i - 1$ edges; for $i = n$ this proves the forward implication. Conversely, let $G$ be any connected graph with $n$ vertices and $n - 1$ edges. Let $G'$ be a spanning tree in $G$. Since $G'$ has $n - 1$ edges by the first implication, it follows that $G = G'$. □

**Corollary 1.5.4.** *If $T$ is a tree and $G$ is any graph with $\delta(G) \geqslant |T| - 1$, then $T \subseteq G$, i.e. $G$ has a subgraph isomorphic to $T$.* [9.2.1] [9.2.3]

*Proof.* Find a copy of $T$ in $G$ inductively along its vertex enumeration from Corollary 1.5.2. □

Sometimes it is convenient to consider one vertex of a tree as special; such a vertex is then called the *root* of this tree. A tree $T$ with a fixed root $r$ is a *rooted tree*. Writing $x \leqslant y$ for $x \in rTy$ then defines a partial ordering on $V(T)$, the *tree-order* associated with $T$ and $r$. We shall think of this ordering as expressing 'height': if $x < y$ we say that $x$ lies *below* $y$ in $T$, we call

*root*

*tree-order*

*up/above*
*down/below*

$$\lceil y \rceil := \{ x \mid x \leqslant y \} \quad \text{and} \quad \lfloor x \rfloor := \{ y \mid y \geqslant x \}$$

$\lceil t \rceil$, $\lfloor t \rfloor$

the *down-closure* of $y$ and the *up-closure* of $x$, and so on. A set $X \subseteq V(T)$ that equals its up-closure, i.e. which satisfies $X = \lfloor X \rfloor := \bigcup_{x \in X} \lfloor x \rfloor$, is *closed upwards*, or an *up-set* in $T$. Similarly, there are *down-closed* sets, or *down-sets* etc..

*down-closure*
*up-closure*

Note that the root of $T$ is the least element in its tree-order, the leaves are its maximal elements, the ends of any edge of $T$ are comparable, and the down-closure of every vertex is a *chain*, a set of pairwise comparable elements. (Proofs?) The vertices at distance $k$ from the root have *height* $k$ and form the $k$th *level* of $T$.

*chain*

*height, level*

A rooted tree $T$ contained in a graph $G$ is called *normal* in $G$ if the ends of every $T$-path in $G$ are comparable in the tree-order of $T$. If $T$ spans $G$, this amounts to requiring that two vertices of $T$ must be comparable whenever they are adjacent in $G$; see Figure 1.5.2.

*normal tree*

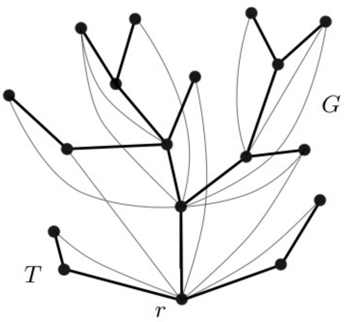


*Fig. 1.5.2.* A normal spanning tree with root $r$

A normal tree $T$ in $G$ can be a powerful tool for examining the structure of $G$, because $G$ reflects the separation properties of $T$:

[8.2.3] [8.5.6] **Lemma 1.5.5.** *Let $T$ be a normal tree in $G$.*

(i) *Any two vertices $x, y \in T$ are separated in $G$ by the set $\lceil x \rceil \cap \lceil y \rceil$.*

(ii) *If $S \subseteq V(T) = V(G)$ and $S$ is down-closed, then the components of $G - S$ are spanned by the sets $\lfloor x \rfloor$ with $x$ minimal in $T - S$.*

*Proof.* (i) Let $P$ be any $x$–$y$ path in $G$; we show that $P$ meets $\lceil x \rceil \cap \lceil y \rceil$. Let $t_1, \ldots, t_n$ be a minimal sequence of vertices in $P \cap T$ such that $t_1 = x$ and $t_n = y$ and $t_i$ and $t_{i+1}$ are comparable in the tree-order of $T$ for all $i$. (Such a sequence exists: the set of *all* vertices in $P \cap T$, in their natural order as they occur on $P$, has this property because $T$ is normal and every segment $t_i P t_{i+1}$ is either an edge of $T$ or a $T$-path.) In our minimal sequence we cannot have $t_{i-1} < t_i > t_{i+1}$ for any $i$, since $t_{i-1}$ and $t_{i+1}$ would then be comparable, and deleting $t_i$ would yield a smaller such sequence. Thus, our sequence has the form

$$x = t_1 > \ldots > t_k < \ldots < t_n = y$$

for some $k \in \{1, \ldots, n\}$. As $t_k \in \lceil x \rceil \cap \lceil y \rceil \cap V(P)$, our proof is complete.

(ii) Consider a component $C$ of $G - S$, and let $x$ be a minimal element of its vertex set. Then $V(C)$ has no other minimal element $x'$: as $x$ and $x'$ would be incomparable, any $x$–$x'$ path in $C$ would by (i) contain a vertex below both, contradicting their minimality in $V(C)$. Hence as every vertex of $C$ lies above some minimal element of $V(C)$, it lies above $x$. Conversely, every vertex $y \in \lfloor x \rfloor$ lies in $C$, for since $S$ is down-closed, the ascending path $xTy$ lies in $T - S$. Thus, $V(C) = \lfloor x \rfloor$.

Let us show that $x$ is minimal not only in $V(C)$ but also in $T - S$. The vertices below $x$ form a chain $\lceil t \rceil$ in $T$. As $t$ is a neighbour of $x$, the maximality of $C$ as a component of $G - S$ implies that $t \in S$, giving $\lceil t \rceil \subseteq S$ since $S$ is down-closed. This completes the proof that every component of $G - S$ is spanned by a set $\lfloor x \rfloor$ with $x$ minimal in $T - S$.

Conversely, if $x$ is any minimal element of $T - S$, it is clearly also minimal in the component $C$ of $G - S$ to which it belongs. Then $V(C) = \lfloor x \rfloor$ as before, i.e., $\lfloor x \rfloor$ spans this component. □

Normal spanning trees are also called *depth-first search trees*, because of the way they arise in computer searches on graphs (Exercise 22). This fact is often used to prove their existence, which can also be shown some very short and clever induction (Exercise 21). The following constructive proof, however, illuminates better how normal trees capture the structure of their host graphs.

[6.5.3] [8.2.4] **Proposition 1.5.6.** *Every connected graph contains a normal spanning tree, with any specified vertex as its root.*

*Proof.* Let $G$ be a connected graph and $r \in G$ any specified vertex. Let $T$ be a maximal normal tree with root $r$ in $G$; we show that $V(T) = V(G)$.

Suppose not, and let $C$ be a component of $G - T$. As $T$ is normal, $N(C)$ is a chain in $T$. Let $x$ be its greatest element, and let $y \in C$ be adjacent to $x$. Let $T'$ be the tree obtained from $T$ by joining $y$ to $x$; the tree-order of $T'$ then extends that of $T$. We shall derive a contradiction by showing that $T'$ is also normal in $G$.

Let $P$ be a $T'$-path in $G$. If the ends of $P$ both lie in $T$, then they are comparable in the tree-order of $T$ (and hence in that of $T'$), because then $P$ is also a $T$-path and $T$ is normal in $G$ by assumption. If not, then $y$ is one end of $P$, so $P$ lies in $C$ except for its other end $z$, which lies in $N(C)$. Then $z \leqslant x$, by the choice of $x$. For our proof that $y$ and $z$ are comparable it thus suffices to show that $x < y$, i.e. that $x \in rT'y$. This, however, is clear since $y$ is a leaf of $T'$ with neighbour $x$. □

## 1.6 Bipartite graphs

Let $r \geqslant 2$ be an integer. A graph $G = (V, E)$ is called *$r$-partite* if $V$ admits a partition into $r$ classes such that every edge has its ends in different classes: vertices in the same partition class must not be adjacent. Instead of '2-partite' one usually says *bipartite*. *r-partite* *bipartite*

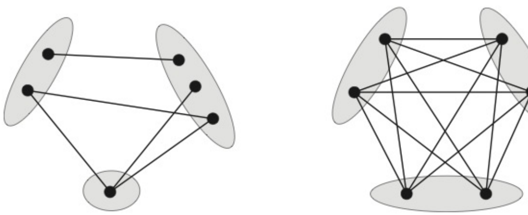


*Fig. 1.6.1.* Two 3-partite graphs

An $r$-partite graph in which *every* two vertices from different partition classes are adjacent is called *complete*; the complete $r$-partite graphs for all $r$ together are the *complete multipartite* graphs. The complete $r$-partite graph $\overline{K^{n_1}} * \ldots * \overline{K^{n_r}}$ is denoted by $K_{n_1,\ldots,n_r}$; if $n_1 = \ldots = n_r =: s$, we abbreviate this to $K_s^r$. Thus, $K_s^r$ is the complete $r$-partite graph in which every partition class contains exactly $s$ vertices.[6] (Figure 1.6.1 shows the example of the octahedron $K_2^3$; compare its drawing with that in Figure 1.4.3.) Graphs of the form $K_{1,n}$ are *complete $r$-partite* $K_{n_1,\ldots,n_r}$ $K_s^r$

[6] Note that we obtain a $K_s^r$ if we replace each vertex of a $K^r$ by an independent $s$-set; our notation of $K_s^r$ is intended to hint at this connection.

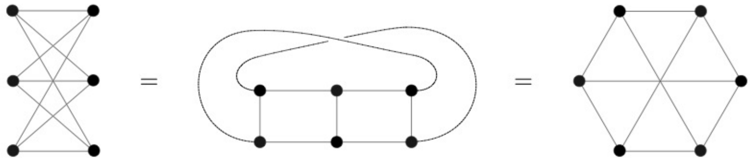


*Fig. 1.6.2.* Three drawings of the bipartite graph $K_{3,3} = K_3^2$

*star* *centre* called *stars*; the vertex in the singleton partition class of this $K_{1,n}$ is the star's *centre*.

*odd cycle* Clearly, a bipartite graph cannot contain an *odd cycle*, a cycle of odd length. In fact, the bipartite graphs are characterized by this property:

[1.9.4] [5.3.1] [6.4.2] **Proposition 1.6.1.** *A graph is bipartite if and only if it contains no odd cycle.*

(1.5.1) *Proof.* Let $G = (V, E)$ be a graph without odd cycles; we show that $G$ is bipartite. Clearly a graph is bipartite if all its components are bipartite or trivial, so we may assume that $G$ is connected. Let $T$ be a spanning tree in $G$, pick a root $r \in T$, and denote the associated tree-order on $V$ by $\leqslant_T$. For each $v \in V$, the unique path $rTv$ has odd or even length. This defines a bipartition of $V$; we show that $G$ is bipartite with this partition.

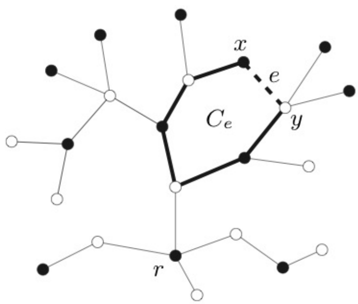


*Fig. 1.6.3.* The cycle $C_e$ in $T + e$

Let $e = xy$ be an edge of $G$. If $e \in T$, with $x <_T y$ say, then $rTy = rTxy$ and so $x$ and $y$ lie in different partition classes. If $e \notin T$ then $C_e := xTy + e$ is a cycle (Fig. 1.6.3), and by the case treated already the vertices along $xTy$ alternate between the two classes. Since $C_e$ is even by assumption, $x$ and $y$ again lie in different classes. □

## 1.7 Contraction and minors

In Section 1.1 we saw two fundamental containment relations between graphs: the 'subgraph' relation, and the 'induced subgraph' relation. In this section we meet two more: the 'minor' relation, and the 'topological minor' relation. Let $X$ be a fixed graph.

A *subdivision* of $X$ is, informally, any graph obtained from $X$ by 'subdividing' some or all of its edges by drawing new vertices on those edges. In other words, we replace some edges of $X$ with new paths between their ends, so that none of these paths has an inner vertex in $V(X)$ or on another new path. When $G$ is a subdivision of $X$, we also say that $G$ *is a* $TX$.[7] The original vertices of $X$ are the *branch vertices of the* $TX$; its new vertices are called *subdividing vertices*. Note that subdividing vertices have degree 2, while branch vertices retain their degree from $X$.

*subdivision* *$TX$ of $X$*

*branch vertices*

If a graph $Y$ contains a $TX$ as a subgraph, then $X$ is a *topological minor* of $Y$ (Fig. 1.7.1).

*topological minor*

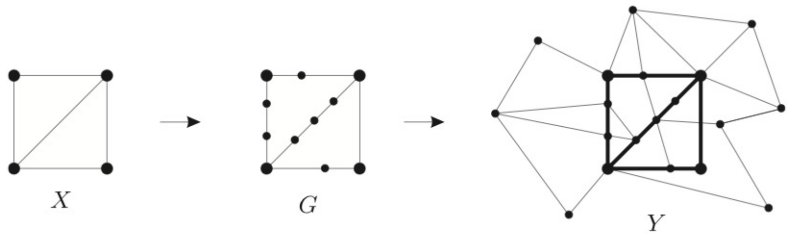


*Fig. 1.7.1.* The graph $G$ is a $TX$, a *subdivision* of $X$. As $G \subseteq Y$, this makes $X$ a *topological minor* of $Y$.

Similarly, replacing the vertices $x$ of $X$ with disjoint connected graphs $G_x$, and the edges $xy$ of $X$ with non-empty sets of $G_x$–$G_y$ edges, yields a graph that we shall call an $IX$.[8] More formally, a graph $G$ *is an* $IX$ if its vertex set admits a partition $\{ V_x \mid x \in V(X) \}$ into connected subsets $V_x$ such that distinct vertices $x, y \in X$ are adjacent in $X$ if and only if $G$ contains a $V_x$–$V_y$ edge. The sets $V_x$ are the *branch sets of the* $IX$. Conversely, we say that $X$ arises from $G$ by *contracting* the subgraphs $G_x$.

*$IX$*

*branch sets*

If a graph $Y$ contains an $IX$ as a subgraph, then $X$ is a *minor* of $Y$ and we write $X \preccurlyeq Y$ (Fig. 1.7.2). Thus, $X$ is a minor of $Y$ if and only if

*minor* $\preccurlyeq$

[7] The '$T$' stands for 'topological'. Although, formally, $TX$ denotes a whole class of graphs, the class of all subdivisions of $X$, it is customary to use the expression as indicated to refer to an arbitrary member of that class.

[8] The '$I$' stands for 'inflated'. As before, while $IX$ is formally a class of graphs, those admitting a vertex partition $\{ V_x \mid x \in V(X) \}$ as described below, we use the expression as indicated to refer to an arbitrary member of that class.

there is a map $\varphi$ from a subset of $V(Y)$ onto $V(X)$ such that for every vertex $x \in X$ its inverse image $\varphi^{-1}(x)$ is connected in $Y$ and for every edge $xx' \in X$ there is an edge in $Y$ between the sets $\varphi^{-1}(x)$ and $\varphi^{-1}(x')$. (In our earlier terminology, these are the branch sets $V_x$ and $V_{x'}$.) If the domain of $\varphi$ is all of $V(Y)$, and $xx' \in X$ whenever $x \neq x'$ and $Y$ has an edge between $\varphi^{-1}(x)$ and $\varphi^{-1}(x')$ (so that $Y$ is an $IX$), we call $\varphi$ a *contraction* of $Y$ onto $X$.

*contraction*

Since branch sets can be singletons, every subgraph of a graph is also its minor. In infinite graphs, branch sets are allowed to be infinite. For example, the graph shown in Figure 8.1.1 is an $IX$ with $X$ an infinite star.

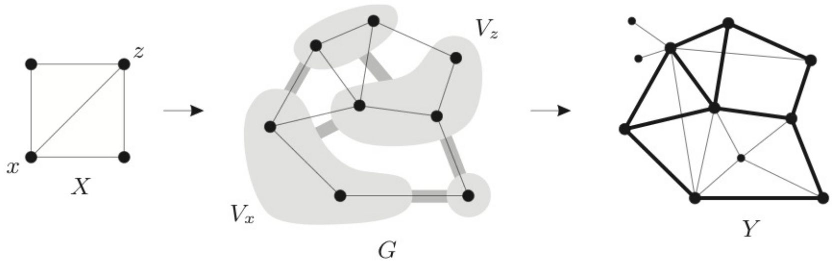


*Fig. 1.7.2.* The graph $G$ is an $IX$, and $G$ is a subgraph of $Y$. This makes $X$ a *minor* of $Y$.

[12.4.1] **Proposition 1.7.1.** *The minor relation $\preccurlyeq$ and the topological-minor relation are partial orderings on the class of finite graphs, i.e. they are reflexive, antisymmetric and transitive.* □

If $G$ is an $IX$ with a branch set $U = V_x$, and every other branch set contains just one vertex, we also write $G/U$ for the graph $X$ and $v_U$ for the vertex $x \in X$ to which $U$ contracts; we then think of the rest of $X$ as an induced subgraph of $G$. The 'smallest' non-trivial case of this is that $U$ contains exactly two vertices forming an edge $e$, so that $U = e$. We then say that $X = G/e$ arises from $G$ by *contracting the edge* $e$; see Figure 1.7.3.

*$G/U$*
*$v_U$*
*contracting an edge*

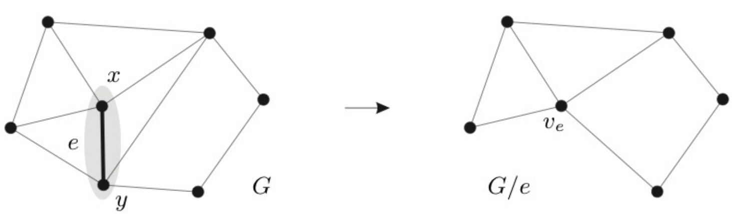


*Fig. 1.7.3.* Contracting the edge $e = xy$

**Proposition 1.7.2.** *A finite graph $G$ is an $IX$ if and only if $X$ can be obtained from $G$ by a sequence of edge contractions, i.e. if and only if there are graphs $G_0, \dots, G_n$ and edges $e_i \in G_i$ such that $G_0 = G$, $G_n \simeq X$, and $G_{i+1} = G_i/e_i$ for all $i < n$.*

*Proof.* Induction on $|G| - |X|$. □

Propositions 1.7.1 and 1.7.2 together imply the following:

**Corollary 1.7.3.** *Let $X$ and $Y$ be finite graphs. $X$ is a minor of $Y$ if and only if there are graphs $G_0, \dots, G_n$ such that $G_0 = Y$ and $G_n = X$ and each $G_{i+1}$ arises from $G_i$ by deleting an edge, contracting an edge, or deleting a vertex.* □

Finally, we have the following relationship between minors and topological minors:

**Proposition 1.7.4.** [4.4.2] [7.3.1] [12.5.3]

(i) *Every $TX$ is also an $IX$ (Fig. 1.7.4); thus, every topological minor of a graph is also its (ordinary) minor.*

(ii) *If $\Delta(X) \leqslant 3$, then every $IX$ contains a $TX$; thus, every minor with maximum degree at most 3 of a graph is also its topological minor.* □

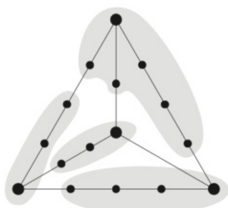

*Fig. 1.7.4.* A subdivision of $K^4$ viewed as an $IK^4$

Now that we have met all the standard relations between graphs, we can also define what it means to embed one graph in another. Basically, an *embedding* of $G$ in $H$ is an injective map $\varphi\colon V(G) \to V(H)$ that preserves the kind of structure we are interested in. Thus, $\varphi$ embeds $G$ in $H$ 'as a subgraph' if it preserves the adjacency of vertices, and 'as an induced subgraph' if it preserves both adjacency and non-adjacency. If $\varphi$ is defined on $E(G)$ as well as on $V(G)$ and maps the edges $xy$ of $G$ to independent paths in $H$ between $\varphi(x)$ and $\varphi(y)$, it embeds $G$ in $H$ 'as a topological minor'. Similarly, an embedding $\varphi$ of $G$ in $H$ 'as a minor' would be a map from $V(G)$ to disjoint connected vertex sets in $H$ (rather than to single vertices) so that $H$ has an edge between the sets $\varphi(x)$ and $\varphi(y)$ whenever $xy$ is an edge of $G$. Further variants are *embedding* possible; depending on the context, one may wish to define embeddings 'as a spanning subgraph', 'as an induced minor' and so on, in the obvious way.

## 1.8 Euler tours

Any mathematician who happens to find himself in the East Prussian city of Königsberg (and in the 18th century) will lose no time to follow the great Leonhard Euler's example and inquire about a round trip through the old city that traverses each of the bridges shown in Figure 1.8.1 exactly once.

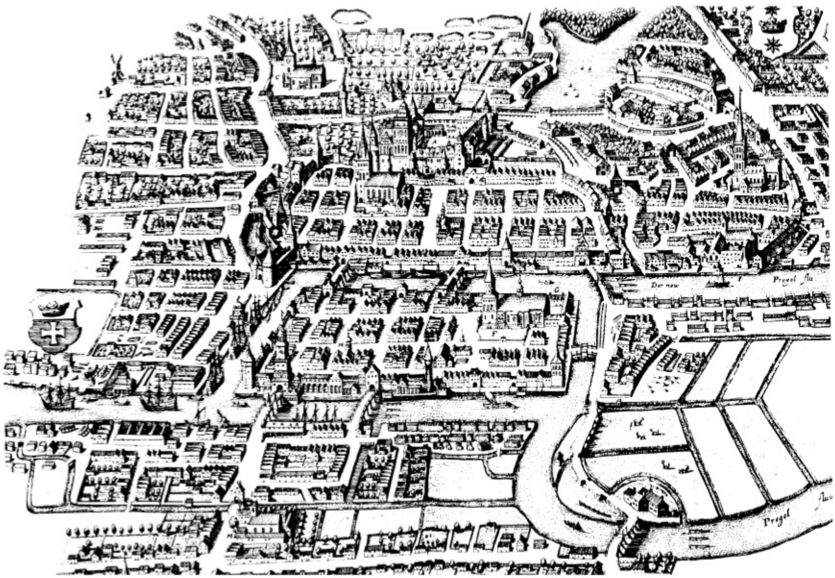

*Fig. 1.8.1.* The bridges of Königsberg (anno 1736)

Thus inspired,[9] let us call a closed walk in a graph an *Euler tour* if it traverses every edge of the graph exactly once. A graph is *Eulerian* if it admits an Euler tour.

*Eulerian*

[2.1.5] [10.3.1]

**Theorem 1.8.1.** (Euler 1736)
*A connected graph is Eulerian if and only if every vertex has even degree.*

*Proof.* The degree condition is clearly necessary: a vertex appearing $k$ times in an Euler tour (or $k+1$ times, if it is the starting and finishing vertex and as such counted twice) must have degree $2k$.

[9] Anyone to whom such inspiration seems far-fetched, even after contemplating Figure 1.8.2, may seek consolation in the *multigraph* of Figure 1.10.1.

Conversely, let $G$ be a connected graph with all degrees even, and let

$$W = v_0 e_0 \dots e_{\ell-1} v_\ell$$

be a longest walk in $G$ using no edge more than once. Since $W$ cannot be extended, it already contains all the edges at $v_\ell$. By assumption, the number of such edges is even. Hence $v_\ell = v_0$ (why?), so $W$ is a closed walk.

Suppose $W$ is not an Euler tour. Then $G$ has an edge $e$ outside $W$ but incident with a vertex of $W$, say $e = uv_i$. (Here we use the connectedness of $G$, as in the proof of Proposition 1.4.1.) Then the walk

$$uev_i e_i \dots e_{\ell-1} v_\ell e_0 \dots e_{i-1} v_i$$

is longer than $W$, a contradiction. □

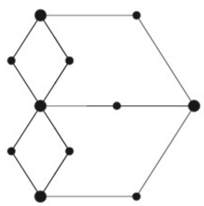

*Fig. 1.8.2.* A graph formalizing the bridge problem

## 1.9 Some linear algebra

Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges, say $V = \{v_1, \dots, v_n\}$ and $E = \{e_1, \dots, e_m\}$. The *vertex space* $\mathcal{V}(G)$ of $G$ is the vector space over the 2-element field $\mathbb{F}_2 = \{0, 1\}$ of all functions $V \to \mathbb{F}_2$. Every element of $\mathcal{V}(G)$ corresponds naturally to a subset of $V$, the set of those vertices to which it assigns a 1, and every subset of $V$ is uniquely represented in $\mathcal{V}(G)$ by its indicator function. We may thus think of $\mathcal{V}(G)$ as the power set of $V$ made into a vector space: the sum $U + U'$ of two vertex sets $U, U' \subseteq V$ is their symmetric difference (why?), and $U = -U$ for all $U \subseteq V$. The zero in $\mathcal{V}(G)$, viewed in this way, is the empty (vertex) set $\emptyset$. Since $\{\{v_1\}, \dots, \{v_n\}\}$ is a basis of $\mathcal{V}(G)$, its *standard basis*, we have $\dim \mathcal{V}(G) = n$.

*$G = (V, E)$*

*vertex space $\mathcal{V}(G)$*

*$+$*

In the same way as above, the functions $E \to \mathbb{F}_2$ form the *edge space* $\mathcal{E}(G)$ of $G$: its elements correspond to the subsets of $E$, vector addition amounts to symmetric difference, $\emptyset \subseteq E$ is the zero, and $F = -F$ for all $F \subseteq E$. As before, $\{\{e_1\}, \dots, \{e_m\}\}$ is the *standard basis* of $\mathcal{E}(G)$, and

*edge space $\mathcal{E}(G)$*

*standard basis*

$\dim \mathcal{E}(G) = m$. Given two elements $F, F'$ of the edge space, viewed as functions $E \to \mathbb{F}_2$, we write

$\langle F, F' \rangle$

$$\langle F, F' \rangle := \sum_{e \in E} F(e)F'(e) \in \mathbb{F}_2 .$$

This is zero if and only if $F$ and $F'$ have an even number of edges in common; in particular, we can have $\langle F, F \rangle = 0$ with $F \neq \emptyset$. Given a subspace $\mathcal{F}$ of $\mathcal{E}(G)$, we write

$\mathcal{F}^\perp$

$$\mathcal{F}^\perp := \{ D \in \mathcal{E}(G) \mid \langle F, D \rangle = 0 \text{ for all } F \in \mathcal{F} \} .$$

This is again a subspace of $\mathcal{E}(G)$ (the space of all vectors solving a certain set of linear equations—which?), and one can show that

$$\dim \mathcal{F} + \dim \mathcal{F}^\perp = m .$$

*cycle space* $\mathcal{C}(G)$

The *cycle space* $\mathcal{C} = \mathcal{C}(G)$ is the subspace of $\mathcal{E}(G)$ spanned by all the cycles in $G$—more precisely, by their edge sets.[10] The dimension of $\mathcal{C}(G)$ is sometimes called the *cyclomatic number* of $G$.

The elements of $\mathcal{C}$ are easily recognized by the degrees of the subgraphs they form. Moreover, to generate the cycle space from cycles we only need disjoint unions rather than arbitrary symmetric differences:

[4.5.1]

**Proposition 1.9.1.** *The following assertions are equivalent for edge sets $F \subseteq E$:*

(i) $F \in \mathcal{C}(G)$;

(ii) *$F$ is a (possibly empty) disjoint union of (edge sets of) cycles in $G$;*

(iii) *All vertex degrees of the graph $(V, F)$ are even.*

*Proof.* Since cycles have even degrees and taking symmetric differences preserves this, (i)→(iii) follows by induction on the number of cycles used to generate $F$. The implication (iii)→(ii) follows by induction on $|F|$: if $F \neq \emptyset$ then $(V, F)$ contains a cycle $C$, whose edges we delete for the induction step. The implication (ii)→(i) is immediate from the definition of $\mathcal{C}(G)$. □

*cut*

A set $F$ of edges is a *cut* in $G$ if there exists a partition $\{V_1, V_2\}$ of $V$ such that $F = E(V_1, V_2)$. The edges in $F$ are said to *cross* this partition. The sets $V_1, V_2$ are the *sides* of the cut. Recall that for $V_1 = \{v\}$ this cut is denoted by $E(v)$.

[10] For simplicity, we shall not always distinguish between the edge sets $F \in \mathcal{E}(G)$ and the subgraphs $(V, F)$ they induce in $G$. When we wish to be more precise, such as in Chapter 8.5, we shall use the word '*circuit*' for the edge set of a cycle.

**Proposition 1.9.2.** *Together with $\emptyset$, the cuts in $G$ form a subspace $\mathcal{C}^*$ of $\mathcal{E}(G)$. This space is generated by cuts of the form $E(v)$.* [4.6.3]

*Proof.* Let $\mathcal{C}^*$ denote the set of all cuts in $G$, together with $\emptyset$. To prove that $\mathcal{C}^*$ is a subspace, we show that for all $D, D' \in \mathcal{C}^*$ also $D + D'$ $(= D - D')$ lies in $\mathcal{C}^*$. Since $D + D = \emptyset \in \mathcal{C}^*$ and $D + \emptyset = D \in \mathcal{C}^*$, we may assume that $D$ and $D'$ are distinct and non-empty. Let $\{V_1, V_2\}$ and $\{V_1', V_2'\}$ be the corresponding partitions of $V$. Then $D + D'$ consists of all the edges that cross one of these partitions but not the other (Fig. 1.9.1). But these are precisely the edges between $(V_1 \cap V_1') \cup (V_2 \cap V_2')$ and $(V_1 \cap V_2') \cup (V_2 \cap V_1')$, and by $D \neq D'$ these two sets form another partition of $V$. Hence $D + D' \in \mathcal{C}^*$, and $\mathcal{C}^*$ is indeed a subspace of $\mathcal{E}(G)$.

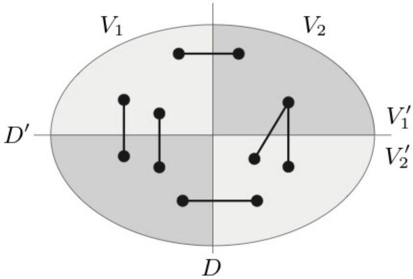


*Fig. 1.9.1.* Cut edges in $D + D'$

Our second assertion, that the cuts $E(v)$ generate all of $\mathcal{C}^*$, follows from the fact that every edge $xy \in G$ lies in exactly two such cuts (in $E(x)$ and in $E(y)$); thus every partition $\{V_1, V_2\}$ of $V$ satisfies $E(V_1, V_2) = \sum_{v \in V_1} E(v)$. $\square$

The subspace $\mathcal{C}^* =: \mathcal{C}^*(G)$ of $\mathcal{E}(G)$ from Proposition 1.9.2 is the *cut space* of $G$. It is not difficult to find among the cuts $E(v)$ an explicit basis for $\mathcal{C}^*(G)$, and thus to determine its dimension (Exercise 34). *cut space* $\mathcal{C}^*(G)$

A minimal non-empty cut in $G$ is a *bond*. Thus, bonds are for $\mathcal{C}^*$ what cycles are for $\mathcal{C}$: the minimal non-empty elements. Note that the 'non-empty' condition bites only if $G$ is disconnected. If $G$ is connected, its bonds are just its minimal cuts, and these are easy to recognize: clearly, a cut in a connected graph is minimal if and only if both sides of the corresponding vertex partition induce connected subgraphs. If $G$ is disconnected, its bonds are the minimal cuts of its components. (See also Lemma 3.1.2.) *bond*

In analogy to Proposition 1.9.1, bonds and disjoint unions suffice to generate $\mathcal{C}^*$:

**Lemma 1.9.3.** *Every cut is a disjoint union of bonds.* [4.6.2] [6.5.2]

*Proof.* We apply induction on the size of the cut $F$ considered. For $F = \emptyset$ the assertion is trivial (with the empty union). If $F \neq \emptyset$ is not itself a bond, it properly contains some other non-empty cut $F'$. By Proposition 1.9.2, also $F \smallsetminus F' = F + F'$ is a smaller non-empty cut. By the induction hypothesis, both $F'$ and $F \smallsetminus F'$ are disjoint unions of bonds, and hence so is $F$. □

Exercise 33 indicates how to construct the bonds for Lemma 1.9.3 explicitly. In Chapter 3.1 we shall prove some more details about the possible positions of the cycles and bonds of a graph within its overall structure (Lemmas 3.1.2 and 3.1.3).

[4.6] **Theorem 1.9.4.** *The cycle space* $\mathcal{C}$ *and the cut space* $\mathcal{C}^*$ *of any graph satisfy*

$$\mathcal{C} = \mathcal{C}^{*\perp} \quad \text{and} \quad \mathcal{C}^* = \mathcal{C}^{\perp}\,.$$

(1.6.1)
(1.10)

*Proof.* Consider a graph $G = (V, E)$. Clearly, any cycle in $G$ has an even number of edges in each cut. This implies $\mathcal{C} \subseteq \mathcal{C}^{*\perp}$ and $\mathcal{C}^* \subseteq \mathcal{C}^{\perp}$.

To prove $\mathcal{C}^{*\perp} \subseteq \mathcal{C}$, recall from Proposition 1.9.1 that for every edge set $F \notin \mathcal{C}$ there exists a vertex $v$ incident with an odd number of edges in $F$. Then $\langle E(v), F \rangle = 1$, so $E(v) \in \mathcal{C}^*$ implies $F \notin \mathcal{C}^{*\perp}$. This completes the proof of $\mathcal{C} = \mathcal{C}^{*\perp}$.

To prove $\mathcal{C}^{\perp} \subseteq \mathcal{C}^*$, let $F \in \mathcal{C}^{\perp}$ be given. Consider the multigraph[11] $H$ obtained by contracting the components of $(V, E \smallsetminus F)$ in $G$. Any cycle in $H$ has all its edges in $F$. Since we can extend it to a cycle in $G$, the number of these edges must be even. Hence $H$ is bipartite, by Proposition 1.6.1. Its bipartition induces a bipartition $(V_1, V_2)$ of $V$ such that $E(V_1, V_2) = F$, showing $F \in \mathcal{C}^*$ as desired. □

Consider a connected graph $G = (V, E)$ with a spanning tree $T \subseteq G$. Recall that for every chord $e \in E \smallsetminus E(T)$ there is a unique cycle $C_e$ in $T + e$ (Fig. 1.6.3); these cycles $C_e$ are the *fundamental cycles* of $G$ with respect to $T$. On the other hand, given an edge $e \in T$, the graph $T - e$ has exactly two components (Theorem 1.5.1 (iii)), and the set $D_e \subseteq E$ of edges between these two components form a bond in $G$ (Fig.1.9.2). These bonds $D_e$ are the *fundamental cuts* of $G$ with respect to $T$.

*fundamental cycles*

(1.5.1)

*fundamental cuts*

[4.5.1] **Theorem 1.9.5.** *Let* $G$ *be a connected graph and* $T \subseteq G$ *a spanning tree. Then the corresponding fundamental cycles and cuts form a basis of* $\mathcal{C}(G)$ *and of* $\mathcal{C}^*(G)$*, respectively. If* $G$ *has* $n$ *vertices and* $m$ *edges, then*

$$\dim \mathcal{C}(G) = m - n + 1 \quad \text{and} \quad \dim \mathcal{C}^*(G) = n - 1\,.$$

[11] See Section 1.10. The proof of Proposition 1.6.1 works for multigraphs too.

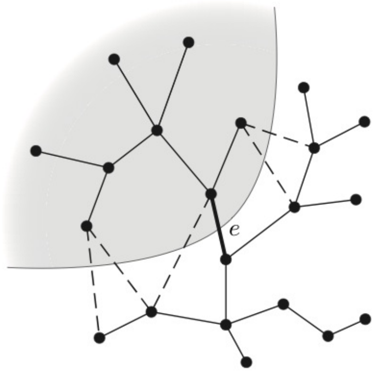


*Fig. 1.9.2.* The fundamental cut $D_e$

*Proof.* Note that an edge $e \in T$ lies in $D_e$ but in no other fundamental cut, while an edge $e \notin T$ lies in $C_e$ but in no other fundamental cycle. Hence, the fundamental cycles and the fundamental cuts form linearly independent sets in $\mathcal{C} = \mathcal{C}(G)$ and $\mathcal{C}^* = \mathcal{C}^*(G)$, respectively. (1.5.3)

Let us show that the fundamental cycles generate every cycle $C$. By our initial observation, $F := C + \sum_{e \in C \setminus T} C_e$ is an element of $\mathcal{C}$ that contains no edge outside $T$. But the only element of $\mathcal{C}$ contained in $T$ is $\emptyset$. So $F = \emptyset$, giving $C = \sum_{e \in C \setminus T} C_e$.

Similarly, every cut $D$ is a sum of fundamental cuts. Indeed, the element $D + \sum_{e \in D \cap T} D_e$ of $\mathcal{C}^*$ contains no edge of $T$. As $\emptyset$ is the only element of $\mathcal{C}^*$ missing $T$, this implies $D = \sum_{e \in D \cap T} D_e$.

So the fundamental cycles and cuts form bases of $\mathcal{C}$ and of $\mathcal{C}^*$. By Corollary 1.5.3 there are $n-1$ fundamental cuts, so there are $m-n+1$ fundamental cycles. □

*incidence matrix*

The *incidence matrix* $B = (b_{ij})_{n \times m}$ of a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$ is defined over $\mathbb{F}_2$ by

$$b_{ij} := \begin{cases} 1 & \text{if } v_i \in e_j \\ 0 & \text{otherwise.} \end{cases}$$

As usual, let $B^t$ denote the transpose of $B$. Then $B$ and $B^t$ define linear maps $B: \mathcal{E}(G) \to \mathcal{V}(G)$ and $B^t: \mathcal{V}(G) \to \mathcal{E}(G)$ with respect to the standard bases.

**Proposition 1.9.6.**

(i) *The kernel of* $B$ *is* $\mathcal{C}(G)$.

(ii) *The image of* $B^t$ *is* $\mathcal{C}^*(G)$. □

*adjacency matrix*

The *adjacency matrix* $A = (a_{ij})_{n \times n}$ of $G$ is defined by

$$a_{ij} := \begin{cases} 1 & \text{if } v_i v_j \in E \\ 0 & \text{otherwise.} \end{cases}$$

Our last proposition establishes a simple connection between $A$ and $B$ (now viewed as real matrices). Let $D$ denote the real diagonal matrix $(d_{ij})_{n\times n}$ with $d_{ii} = d(v_i)$ and $d_{ij} = 0$ otherwise.

**Proposition 1.9.7.** $BB^t = A + D$. □

## 1.10 Other notions of graphs

For completeness, we now mention a few other notions of graphs which feature less frequently or not at all in this book.

*hypergraph* A *hypergraph* is a pair $(V, E)$ of disjoint sets, where the elements of $E$ are non-empty subsets (of any cardinality) of $V$. Thus, graphs are special hypergraphs.

*directed graph* A *directed graph* (or *digraph*) is a pair $(V, E)$ of disjoint sets (of *vertices* and *edges*) together with two maps $\text{init}\colon E \to V$ and $\text{ter}\colon E \to V$ assigning to every edge $e$ an *initial vertex* $\text{init}(e)$ and a *terminal vertex* $\text{ter}(e)$. The edge $e$ is said to be *directed from* $\text{init}(e)$ *to* $\text{ter}(e)$. Note that a directed graph may have several edges between the same two vertices $x, y$. Such edges are called *multiple edges*; if they have the same direction (say from $x$ to $y$), they are *parallel*. If $\text{init}(e) = \text{ter}(e)$, the edge $e$ is called a *loop*.

init($e$)
ter($e$)
*loop*

*orientation* A directed graph $D$ is an *orientation* of an (undirected) graph $G$ if $V(D) = V(G)$ and $E(D) = E(G)$, and if $\{\text{init}(e), \text{ter}(e)\} = \{x, y\}$ for every edge $e = xy$. Intuitively, such an *oriented graph* arises from an undirected graph simply by directing every edge from one of its ends to the other. Put differently, oriented graphs are directed graphs without loops or multiple edges.

*oriented graph*

*multigraph* A *multigraph* is a pair $(V, E)$ of disjoint sets (of *vertices* and *edges*) together with a map $E \to V \cup [V]^2$ assigning to every edge either one or two vertices, its *ends*. Thus, multigraphs too can have loops and multiple edges: we may think of a multigraph as a directed graph whose edge directions have been 'forgotten'. To express that $x$ and $y$ are the ends of an edge $e$ we still write $e = xy$, though this no longer determines $e$ uniquely.

A graph is thus essentially the same as a multigraph without loops or multiple edges. Somewhat surprisingly, proving a graph theorem more generally for multigraphs may, on occasion, simplify the proof. Moreover, there are areas in graph theory (such as plane duality; see Chapters 4.6 and 6.5) where multigraphs arise more naturally than graphs, and where any restriction to the latter would seem artificial and be technically complicated. We shall therefore consider multigraphs in these cases, but without much technical ado: terminology introduced earlier for graphs will be used correspondingly.

A few differences, however, should be pointed out. A multigraph may have cycles of length 1 or 2: loops, and pairs of multiple edges (or *double edges*). A loop at a vertex makes it its own neighbour, and contributes 2 to its degree; in Figure 1.10.1, we thus have $d(v_e) = 6$. The ends of loops and parallel edges in a multigraph $G$ are considered as separating that edge from the rest of $G$. The vertex $v$ of a loop $e$, therefore, is a cutvertex unless $(\{v\}, \{e\})$ is a component of $G$, and $(\{v\}, \{e\})$ is a 'block' in the sense of Chapter 3.1. Thus, a multigraph with a loop is never 2-connected, and any 3-connected multigraph is in fact a graph.

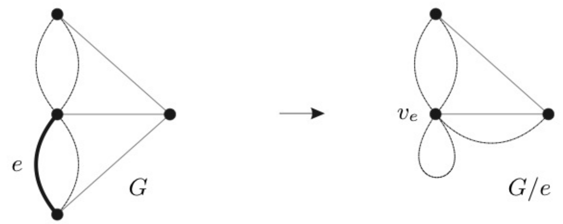


*Fig. 1.10.1.* Contracting the edge $e$ in the multigraph corresponding to Fig. 1.8.1

The notion of edge contraction is simpler in multigraphs than in graphs. If we contract an edge $e = xy$ in a multigraph $G = (V, E)$ to a new vertex $v_e$, there is no longer a need to delete any edges other than $e$ itself: edges parallel to $e$ become loops at $v_e$, while edges $xv$ and $yv$ become parallel edges between $v_e$ and $v$ (Fig. 1.10.1). Thus, formally, $E(G/e) = E \setminus \{e\}$, and only the incidence map $e' \mapsto \{\mathrm{init}(e'), \mathrm{ter}(e')\}$ of $G$ has to be adjusted to the new vertex set in $G/e$. The notion of a minor adapts to multigraphs accordingly. Note that the contraction of loops remains undefined: they can be deleted but not contracted.

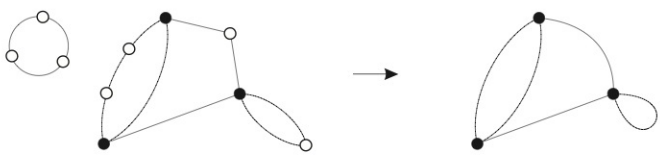

*Fig. 1.10.2.* Suppressing the white vertices

*suppressing a vertex*

If $v$ is a vertex of degree 2 in a multigraph $G$, then by *suppressing* $v$ we mean deleting $v$ and adding an edge between its two neighbours.[12] (If its two incident edges are identical, i.e. form a loop at $v$, we add no edge and obtain just $G - v$. If they go to the same vertex $w \neq v$, the added edge will be a loop at $w$. See Figure 1.10.2.) Since the degrees

[12] This is just a clumsy combinatorial paraphrase of the topological notion of amalgamating the two edges at $v$ into one edge, of which $v$ becomes an inner point.

of all vertices other than $v$ remain unchanged when $v$ is suppressed, suppressing several vertices of $G$ always yields a well-defined multigraph that is independent of the order in which those vertices are suppressed.

Finally, it should be pointed out that authors who usually work with multigraphs tend to call them 'graphs'; in their terminology, our graphs would be called 'simple graphs'.

## Exercises

1.$^-$ What is the number of edges in a $K^n$?

2. Let $d \in \mathbb{N}$ and $V := \{0,1\}^d$; thus, $V$ is the set of all 0–1 sequences of length $d$. The graph on $V$ in which two such sequences form an edge if and only if they differ in exactly one position is called the *$d$-dimensional cube*. Determine the average degree, number of edges, diameter, girth and circumference of this graph.

   (Hint for the circumference: induction on $d$.)

3. Let $G$ be a graph containing a cycle $C$, and assume that $G$ contains a path of length at least $k$ between two vertices of $C$. Show that $G$ contains a cycle of length at least $\sqrt{k}$.

4.$^-$ Is the bound in Proposition 1.3.2 best possible?

5. Let $v_0$ be a vertex in a graph $G$, and $D_0 := \{v_0\}$. For $n = 1, 2, \dots$ inductively define $D_n := N_G(D_0 \cup \ldots \cup D_{n-1})$. Show that $D_n = \{ v \mid d(v_0, v) = n \}$ and $D_{n+1} \subseteq N(D_n) \subseteq D_{n-1} \cup D_{n+1}$ for all $n \in \mathbb{N}$.

6. Show that $\operatorname{rad} G \leqslant \operatorname{diam} G \leqslant 2 \operatorname{rad} G$ for every graph $G$.

7. Prove the weakening of Theorem 1.3.4 obtained by replacing average with minimum degree. Deduce that $|G| \geqslant n_0(d/2, g)$ for every graph $G$ as given in the theorem.

8.$^+$ Show that every connected graph $G$ contains a path of length at least $\min\{2\delta(G), |G|-1\}$.

9. Show that a connected graph of diameter $k$ and minimum degree $d$ has at least about $kd/3$ vertices but need not have substantially more.

10.$^-$ Show that the components of a graph partition its vertex set. (In other words, show that every vertex belongs to exactly one component.)

11.$^-$ Show that every 2-connected graph contains a cycle.

12. Determine $\kappa(G)$ and $\lambda(G)$ for $G = P^m, C^n, K^n, K_{m,n}$ and the $d$-dimensional cube (Exercise 2); $d, m, n \geqslant 3$.

13.$^-$ Is there a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for all $k \in \mathbb{N}$, every graph of minimum degree at least $f(k)$ is $k$-connected?

14.$^{+}$ Let $\alpha, \beta$ be two graph invariants with positive integer values. Formalize the two statements below, and show that each implies the other:

(i) $\alpha$ is bounded above by a function of $\beta$;

(ii) $\beta$ can be forced up by making $\alpha$ large enough.

Show that the statement

(iii) $\beta$ is bounded below by a function of $\alpha$

is not equivalent to (i) and (ii). Which small change will make it so?

15. Consider the proof of Theorem 1.4.3. Would it not seem more natural to assume in the second statement of $(*)$ that $\varepsilon(G') > \gamma - k$, as required for $H$ in the statement of the theorem?

(i) Look how this alteration would change the proof: which parts would carry over, which could be adapted, and which would fail?

(ii) Explain how the use of an assumption of the form $m \geqslant c_k n - b_k$ rather than $m \geqslant c_k n$ helps to obtain a contradiction in the final inequality of the proof.

16. Prove Theorem 1.5.1.

17.$^{-}$ Show that every tree $T$ has at least $\Delta(T)$ leaves.

18. Show that a tree without a vertex of degree 2 has more leaves than other vertices. Can you find a very short proof that does not use induction?

19. Show that the tree-order associated with a rooted tree $T$ is indeed a partial order on $V(T)$, and verify the claims made about this partial order in the text.

20. Show that a graph is 2-edge-connected if and only if it has a *strongly connected* orientation, one in which every vertex can be reached from every other vertex by a directed path.

21.$^{+}$ Find a short inductive proof for the existence of normal spanning trees in finite connected graphs.

22.$^{+}$ Let $G$ be a connected graph, and let $r \in G$ be a vertex. Starting from $r$, move along the edges of $G$, going whenever possible to a vertex not visited so far. If there is no such vertex, go back along the edge by which the current vertex was first reached (unless the current vertex is $r$; then stop). Show that the edges traversed form a normal spanning tree in $G$ with root $r$.

(This procedure has earned those trees the name of *depth-first search* trees.)

23. Let $\mathcal{T}$ be a set of subtrees of a tree $T$, and $k \in \mathbb{N}$.

(i) Show that if the trees in $\mathcal{T}$ have pairwise non-empty intersection then their overall intersection $\bigcap \mathcal{T}$ is non-empty.

(ii) Show that either $\mathcal{T}$ contains $k$ disjoint trees or there is a set of at most $k-1$ vertices of $T$ meeting every tree in $\mathcal{T}$.

24. Show that every automorphism of a tree fixes a vertex or an edge.

25.$^-$ Do the partition classes of a regular bipartite graph always have the same size?

26.$^-$ Show that a graph is bipartite if and only if every *induced* cycle has even length.

27.$^+$ Prove or disprove that a graph is bipartite if and only if no two adjacent vertices have the same distance from any other vertex.

28.$^+$ Find a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for all $k \in \mathbb{N}$, every graph of average degree at least $f(k)$ has a bipartite subgraph of minimum degree at least $k$.

29. Show that the minor relation $\preccurlyeq$ defines a partial ordering on any set of pairwise non-isomorphic finite graphs. Is the same true for infinite graphs?

30. Prove or disprove that every connected graph contains a walk that traverses each of its edges exactly once in each direction.

31.$^-$ Show that a cut in a connected graph $G$ is a bond if and only if both parts of the corresponding bipartition of $V(G)$ are connected in $G$.

32. Show that the cycle space of a graph is spanned by

    (i) its induced cycles;

    (ii) its geodetic cycles.

    (A cycle $C \subseteq G$ is *geodetic* in $G$ if, for every two vertices of $C$, their distances in $G$ equals their distance in $C$.)

33. Let $F$ be a cut in $G$, with vertex partition $\{V_1, V_2\}$. For $i = 1, 2$ let $C_1^i, \dots, C_{k(i)}^i$ denote the components of $G[V_i]$. Use the $C_j^i$ to define bonds whose disjoint union is $F$.

34. Given a graph $G$, find among all cuts of the form $E(v)$ a basis for the cut space of $G$.

35. Prove that the cycles and the cuts in a graph together generate its entire edge space, or find a counterexample.

36.$^-$ Show the following duality relationship between the fundamental cycles $C_e$ and the fundamental cuts $D_f$ in a graph with a fixed spanning tree: $e \in D_f \Leftrightarrow f \in C_e$.

37. Show that in a connected graph the minimal edge sets containing an edge from every spanning tree are precisely its bonds.

38. Let $F$ be a set of edges in a graph $G$.

    (i) Show that $F$ extends to an element of $\mathcal{C}^*(G)$ if and only if it contains no odd cycle.

    (ii)$^+$ Show that $F$ extends to an element of $\mathcal{C}(G)$ if and only if it contains no odd cut.

39.$^+$ Prove Gallai's theorem that the edge set of any graph $G$ can be written as a disjoint union $E(G) = C \cup D$ with $C \in \mathcal{C}(G)$ and $D \in \mathcal{C}^*(G)$.

40. Show that a set of vertices lies in the image of the incidence matrix of a connected graph if and only if it has even cardinality.

41. Let $A = (a_{ij})_{n\times n}$ be the adjacency matrix of the graph $G$. Show that the matrix $A^k = (a'_{ij})_{n\times n}$ displays, for all $i, j \leqslant n$, the number $a'_{ij}$ of walks of length $k$ from $v_i$ to $v_j$ in $G$.

## Notes

The terminology used in this book is mostly standard. Alternatives do exist, of course, and some of these are stated when a concept is first defined. There is one small point where our notation deviates slightly from standard usage. Complete graphs, paths, cycles etc. of given order are usually denoted by $K_n$, $P_k$, $C_\ell$ and so on, but we use superscripts instead of subscripts. This has the advantage of leaving the variables $K$, $P$, $C$ etc. free for ad-hoc use: we may now enumerate components as $C_1, C_2, \ldots$, speak of paths $P_1, \ldots, P_k$, and so on—without any danger of confusion.

Theorem[13] 1.3.4 was proved by N. Alon, S. Hoory and N. Linial, The Moore bound for irregular graphs, *Graphs Comb.* **18** (2002), 53–57. The proof uses an ingenious argument counting random walks along the edges of the graph considered.

The main assertion of Theorem 1.4.3, that an average degree of at least $4k$ forces a $k$-connected subgraph, is from W. Mader, Existenz $n$-fach zusammenhängender Teilgraphen in Graphen genügend großer Kantendichte, *Abh. Math. Sem. Univ. Hamburg* **37** (1972) 86–97.

The intuition behind the notion of contraction is topological. When $G$ is an $IX$ and we view graphs topologically, we can reobtain $X$ from $G$ by contracting in each of the connected graphs $G[V_x]$ a spanning tree $T_x$ to a single vertex $x$, and deleting any loops or multiple edges that arise in the contraction. This topological background also explains why in multigraphs we may delete loops but not 'contract' them: contraction should not change the homotopy type of a multigraph.

For the history of the Königsberg bridge problem, and Euler's actual part in its solution, see N.L. Biggs, E.K. Lloyd & R.J. Wilson, *Graph Theory 1736–1936*, Oxford University Press 1976.

Of the large subject of algebraic methods in graph theory, Section 1.9 does not convey an adequate impression. A good introduction is N.L. Biggs, *Algebraic Graph Theory* (2nd edn.), Cambridge University Press 1993. A more comprehensive account is given by C.D. Godsil & G.F. Royle, *Algebraic Graph Theory*, Springer GTM 207, 2001. Surveys on the use of algebraic methods can also be found in the *Handbook of Combinatorics* (R.L. Graham, M. Grötschel & L. Lovász, eds.), North-Holland 1995. See also Chung's book cited below.

In algebraic graph theory one usually takes as the elements of the vertex and edge space the functions mapping the vertices, respectively the oriented

[13] In the interest of readability, the end-of-chapter notes in this book give references only for Theorems, and only in cases where these references cannot be found in a monograph or survey cited for that chapter.

edges, to the reals. Then there are $2^m$ standard bases of $\mathcal{E}$ and $2^m$ incidence matrices, one for every choice of edge orientations. (No more, since we require that such functions $\psi$ satisfy $\psi(e,u,v) = -\psi(e,v,u)$ for every pair of inverse orientations of the same edge $e$.) For every fixed choice of orientations, the corresponding incidence matrix represents with respect to the corresponding basis of $\mathcal{E}$ the *boundary map* $\partial\colon \mathcal{E} \to \mathcal{V}$ that assigns to every (basis element for the) oriented edge $(e,u,v)$ the map $V \to \mathbb{R}$ assigning 1 to $v$ and $-1$ to $u$ and 0 to every other vertex (and which extends linearly to all of $\mathcal{E}$). Similarly, the transpose of the incidence matrix represents the *coboundary map* $\delta\colon \mathrm{Hom}(\mathcal{V},\mathbb{R}) \to \mathrm{Hom}(\mathcal{E},\mathbb{R})$ mapping $\varphi$ to $\varphi \circ \partial$; thus, $\delta$ is dual to $\partial$ in the linear algebra sense. The product of the incidence matrix and its transpose is now $BB^t = D - A$, the *Laplacian* of $G$. Note that, unlike $B$, the Laplacian is independent of our choice of basis for $\mathcal{E}$, i.e., of our initial choice of orientations that defined our basis. It plays a fundamental role in algebraic graph theory and its connections to other areas of mathematics; see F.R.K. Chung, *Spectral Graph Theory*, AMS 1997 for much more.